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A SURVEY OF THE THEORY OF SPECTRAL OPERATORS

NELSON DUNFORD

Dedicated to Marston Morse

1. Introduction and statement of the problem. The importance of the spectral reduction theory for bounded and unbounded selfadjoint and normal operators in Hilbert space is amply demonstrated by its diverse applications to such far reaching fields of mathematics as the theories of topological groups, almost periodic functions, harmonic analysis, and selfadjoint boundary value problems. The problems centering around the reduction theory for nonnormal operators are among the most important problems in the theory of linear operators. Notable among the many early contributions to such problems were those of I. Fredholm [20] in 1903 and G. D. Birkhoff [5] in 1908. Fredholm discussed a certain class of linear integral equations and Birkhoff, a class of linear differential boundary value problems on a finite interval. The operators discussed in the Fredholm theory are compact and have spectra which are at worst convergent sequences. The corresponding spectral resolutions need not be countably additive, or, what amounts to the same thing, the eigenvalue expansions need not be unconditionally convergent. The Fredholm theory was later given a more abstract basis, stated in operator form and free of determinant theory, by F. Riesz [32], J. Schauder [33], and T. H. Hildebrandt [22]. The deep and comprehensive work of Birkhoff on eigenvalue expansions associated with (not necessarily selfadjoint) differential operators of arbitrary order strongly suggests that, except for certain irregular cases, linear differential boundary value problems on a finite interval will have unconditionally convergent (in Hilbert space) eigenvalue expansions. That this is indeed the case is shown by the work of J. T. Schwartz[1] and H. P. Kramer [24] who have given Birkhoff's results in an abstract linear operator form. The general formulation shows that the expansion theory is valid for operators whose analytical expressions may involve integral and difference operators as well as other types of terms. The recently announced results of M. A. Neumark [28; 29; 30] on *singular* differ-

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[1] This, and other work of Schwartz referred to here will appear in *Linear operations*, Part II, by N. Dunford and J. T. Schwartz, a forthcoming volume to be published by Interscience Publishers. This volume will be referred to as L.O. II. See also [34].

ential operators of the second order suggests that a countably additive spectral reduction should be expected for a large class of nth order linear differential boundary value problems on the infinite interval. Schwartz has developed and applied operator theory to the extent of being able to give an operator-theoretic basis for many singular nonselfadjoint boundary value problems of the second order and in particular he has established a countably additive spectral reduction for the type of operators considered by Neumark. The analytical difficulties arising in the nth order case have retarded corresponding progress for the general problem. However, the research, as yet unpublished, of David McGarvey indicates that a large class of nth order differential operators with periodic coefficients will have a countably additive spectral reduction. It thus appears that enough evidence exists to justify a serious study, in abstract form, of operators which admit a countably additive spectral resolution. In fact, it has been conjectured by Schwartz that all singular linear differential boundary value problems on an infinite interval I with very mild growth restrictions on the coefficients (except for those of a highly irregular or pathological character similar to the irregular cases of G. D. Birkhoff) determine, in a sense that will be made more precise presently, operators in $L_2(I)$ which have a countably additive spectral resolution.

The main problem, of course, is that of discovering conditions on an operator which, on the one hand, are sufficient to insure the existence of a countably additive resolution of the identity and, on the other hand, are stated in a form that may be applied to the more concrete problems of mathematical study. A beginning has been made on this problem and on many related problems all pertaining to the study of operators with countably additive spectral resolutions defined on the Borel sets in the complex plane. In this lecture I shall try to survey the present state of knowledge concerning such operators, which, for brevity, I shall call spectral operators, and in particular I shall describe briefly some of the applications of the theory of spectral operators to boundary value problems.

A word about the *spectral reduction problem* as opposed to the *general reduction problem* may be in order. The general problem is that of finding all projections E which reduce an operator T i.e., which commute with it. For if E and thus $E'=I-E$ commutes with T then the whole B-space $\mathfrak{X}$ in which T operates is the direct sum of the invariant subspaces $E\mathfrak{X}$ and $E'\mathfrak{X}$ and the study of T is reduced to the study of T on the invariant subspaces. Such a formulation of the reduction problem is not one that will suit our purposes in the study of a *spectral reduction* of an operator. For example, the general problem when applied to the identity operator is that of finding all projections. Whereas, from the spectral point of view, the identity, whose spectrum is a single point, may not be reduced further. A precise formulation of the *spectral reduction problem* for a bounded linear operator is as follows. For each set δ in the family B of Borel sets in the complex plane we wish to find a projection $E(\delta)$ which reduces a given operator T in the complex Banach space $\mathfrak{X}$ and is such that the spectrum of the restriction $T|E(\delta)\mathfrak{X}$ of T to $E(\delta)\mathfrak{X}$ is contained in the closure $\bar{\delta}$ of δ, i.e.,

$$\text{(i)} \qquad TE(\delta) = E(\delta)T, \qquad \sigma(T|E(\delta)\mathfrak{X}) \subseteq \bar{\delta}, \ \delta \in B.$$

The map $\delta \rightarrow E(\delta)$ of the Boolean algebra B into the Boolean algebra of projections $E(\delta)$ should be a homomorphism mapping the units in B into 0 and I, i.e.,

$$\text{(ii)} \quad \begin{array}{lll} E(\phi) = 0, & E(p) = I, & E(\delta') = E(\delta)', \\ E(\delta \cap \sigma) = E(\delta) \wedge E(\sigma), & & E(\delta \cup \sigma) = E(\delta) \vee E(\sigma), \end{array}$$

where we have written ϕ, p for the void set and the whole complex plane respectively, δ' for the complement of δ in p, A' for the *complementary projection* $I-A$, and $A \vee B$, $A \wedge B$ for $A+B-AB$, AB respectively. A third condition demanded of the projections $E(\delta)$ is that they be bounded in δ, i.e.,

$$\text{(iii)} \qquad |E(\delta)| \leqq M, \qquad \delta \in B,$$

for some constant M independent of the Borel set δ. Finally it is required that $E(\delta)$ be countably additive in δ in the strong operator topology, i.e., for every sequence $\{\delta_n\}$ of disjoint Borel sets

$$\text{(iv)} \qquad E\left(\bigcup_{n=1}^{\infty} \delta_n\right)x = \sum_{n=1}^{\infty} E(\delta_n)x, \qquad x \in \mathfrak{X}.$$

(According to a theorem of Orlicz-Pettis [31] this condition of countable additivity will hold if each of the scalar functions $x^*E(\delta)x$ with x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$ is countably additive on B.) The conditions (i), $\cdots$, (iv) are clearly redundant but that does not concern us here. They state in clear form the four basic properties of the map E: $\delta \rightarrow E(\delta)$ which, since it is uniquely determined by T, is called *the resolution of the identity for T* or *the spectral resolution of T*. The spectral reduction problem for an operator T in a complex B-space $\mathfrak{X}$ is thus that of finding a resolution of the identity for T, i.e., a map $\delta \rightarrow E(\delta)$ of the Borel sets δ in the complex plane into a family of projections $E(\delta)$ having the properties (i), $\cdots$, (iv). An operator T which has a spectral resolution in this sense is called a *spectral operator*. It has been observed [11] that the spectral reduction problem is, for finite dimensional spaces equivalent to the canonical reduction of Jordan in classical matrix theory. Thus every operator in a finite dimensional space is a spectral operator. The spectral theorem in Hilbert space shows that every bounded selfadjoint or normal operator is a spectral operator. In the preceding formulation of the spectral reduction problem the operator T was tacitly assumed to be bounded. In case T is merely a closed linear operator with domain $\mathfrak{D}(T)$ in $\mathfrak{X}$ the condition (i) must be replaced by the condition (i)′ that follows.

$$
\text{(i)}' \qquad \begin{cases} \mathfrak{D}(T) \supseteq E(\delta)\mathfrak{X}, & \delta \textit{ bounded}, \\ E(\delta)\mathfrak{D}(T) \subseteq \mathfrak{D}(T), \quad TE(\delta)x = E(\delta)Tx, & x \in \mathfrak{D}(T), \delta \in B, \\ \sigma(T \mid E(\delta)(\mathfrak{X}) \subseteq \bar{\delta}, & \delta \in B, \end{cases}
$$

where the restriction $T|E(\delta)\mathfrak{X}$ of T to $E(\delta)\mathfrak{X}$ has domain $\mathfrak{D}(T) \cap E(\delta)\mathfrak{X}$. As in the bounded case the spectral resolution is uniquely determined[2] by T. Well known examples of unbounded spectral operators are the unbounded selfadjoint operators in Hilbert space.

Besides the finite matrices and the selfadjoint or normal operators in Hilbert space I shall list here a few other examples of bounded and unbounded spectral operators of frequent occurrence in mathematical analysis. K. O. Friedrichs [21] has shown that the operator

$$
\text{(v)} \qquad (Tf)(s) = sf(s) + \int_a^b K(s, t)f(t)dt,
$$

where the kernel K satisfies certain Lipschitz conditions, is actually similar to the selfadjoint operation $(Af)(s) = sf(s)$ in $L_2(a, b)$ and thus T has a spectral resolution. The same result probably holds if T is regarded as an operator in $L_p(a, b)$ with $1 < p < \infty$ but, as far as I know, the details have not been checked. Friedrich suggests in his work that his result is also valid if, instead of assuming the Lipschitz conditions on K, it is assumed that K belongs to a certain class of Fourier transforms. This suggestion was carried out by J. T. Schwartz who found that the operator T of equation (v) is a spectral operator in $L_p(a, b)$ with $1 < p < \infty$ provided that the kernel K is the Fourier transform of a Borel measure μ in the plane whose total variation is less than $(2\pi)^{-1}$. That is, the operator T of equation (v) is a spectral operator in $L_p(a, b)$ with $1 < p < \infty$ provided that

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[2] Bade [1] has developed the theory of unbounded spectral operators. Further results will be found in [15].

$$\text{(vi)} \qquad K(s, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(su+tv)}\mu(du, dv), \qquad \operatorname{var}(\mu) < 1/2\pi.$$

An analogous result is true on the infinite interval. More precisely, if multiplication $(Af)(s) = sf(s)$ is defined on the domain $\mathfrak{D}(A)$ of those f in $L_p(-\infty, \infty)$ for which

$$\int_{-\infty}^{\infty} |sf(s)|^p ds < \infty$$

and if K is a kernel satisfying (vi) and the further restriction that the operations

$$(Kf)(s) = \int_{-\infty}^{\infty} K(s, t)f(t)dt, \qquad (Lf)(t) = \int_{-\infty}^{\infty} K(s, t)f(s)ds$$

are bounded linear operations in $L_p(-\infty, \infty)$ then the operation

$$\text{(vii)} \qquad (Tf)(s) = sf(s) + \int_{-\infty}^{\infty} K(s, t)f(t)dt$$

is, for $1 < p < \infty$, an unbounded spectral operator in $L_p(-\infty, \infty)$ with domain $\mathfrak{D}(A)$ which is similar to the operator A.

The results stated in the preceding paragraph may be proved by Friedrich's methods of establishing similarity of operators. Another method which has been useful in establishing the existence of a spectral resolution for nonselfadjoint operators consists of determining analytical conditions on the resolvent of an operator which are sufficient to insure the existence of projections $E(\delta)$ satisfying conditions (i), $\cdots$, (iv). This method, which will be indicated in some detail in §§3 and 4 that follow, leads to the surprising result that, for large classes of operators which occur among the natural objects of mathematical study, the condition on the resolvent which states the boundedness condition (iii) is by itself sufficient to insure that the operator is a spectral operator. It should be mentioned that the analytical forms taken on by the boundedness condition (iii) in the various applications given explicitly later, are by no means easily applied in all examples. In fact only in the known case of selfadjoint operators in Hilbert space is it clearly satisfied. In all other cases where the condition has been verified the analytical calculations have been considerable. A few examples of differential operators for which Schwartz has verified the boundedness condition are the following. Other examples have been recently discovered by D. R. Smart.[3]

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[3] Communicated to the author by letter.

In the case of differential operators associated with boundary value problems on a finite interval the situation is the following. The unbounded operator T in $L_2(a, b)$ determined by the formal differential expression

$$(Tf)(t) = f^{(n)}(t) + \sum_{j=0}^{n-1} a_j(t)f^{(j)}(t), \tag{viii}$$

where the a_j's are arbitrary bounded, complex, measurable functions, if restricted by a set of linearly independent, homogeneous, and regular (in the sense of Birkhoff) boundary conditions, is a spectral operator in $L_2(a, b)$. In this case the resolvent is compact and thus the spectrum is discrete and only a finite number of the eigenvalues have index[4] greater than one. The associated generalized[5] eigenvalue expansion is unconditionally convergent. All of these properties hold for a somewhat more general class of unbounded operators, e.g., operators T in $L_2(a, b)$ having the form

$$(Tf)(t) = f^{(n)}(t) + \sum_{j=0}^{n-2} B_j f^{(j)} \tag{ix}$$

where the B_j's are arbitrary bounded linear operators in $L_2(a, b)$. The operators B_j may be integral or difference operators or a combination of such and thus there is a large class of linear nth order boundary value problems in a finite interval which determine spectral operators.

As mentioned earlier Neumark has recently announced results which, in our terminology, amount to the assertion that certain singular second order differential operators are spectral operators. Following an idea suggested by Neumark, Schwartz has verified the conditions of the general theory of spectral operators and thus has put the Neumark type of singular differential operator into the framework of general spectral theory. Briefly (and somewhat incompletely) stated the Neumark-Schwartz result (or rather one of the numerous such results) is that the unbounded operator T in $L_2(0, \infty)$ determined by the formal differential operator

$$\tau = -(d/dt)^2 + q(t), \qquad 0 \leqq t < \infty \tag{x}$$

and an arbitrary nontrivial linear homogeneous boundary condition at zero, where q is a bounded complex valued measurable function for which

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[4] The index of an eigenvalue λ is the smallest non-negative integer n for which the equation $(T-\lambda I)^n x = 0$ has the same solutions as the equation $(T-\lambda I)^{n+1}x = 0$.

[5] A generalized eigenvalue expansion is one of the form $x = \sum x_m$ where $(T-\lambda_m I)^{n_m} x_m = 0$.

$$\int_0^\infty (1 + t^2) \, | \, q(t) \, | \, dt < \infty,$$

is a spectral operator. Basically this is a result on the perturbation of spectral operators with continuous spectrum. Closely related results for the selfadjoint case were established by Moser [27]. The main part of the analytical work involved in verifying the conditions required by the general theory is in obtaining sufficiently good asymptotic estimates for the solutions of the equation $\tau f = \lambda f$.

In formulating the spectral reduction problem as we have done by demanding boundedness and countable additivity of the spectral resolution we have admittedly ruled out many interesting and elementary operators. For example Wermer [38] has shown that the map $\{\alpha_n\} \rightarrow \{\alpha_{n+1}\}$ in $l_2(0, \infty)$ is reduced by no bounded projection other than 0 and I. Fixman[6] has recently shown that even the unitary shift operator in $l_p(-\infty, \infty)$ fails, in case $p \neq 2$, to have a countably additive resolution of the identity. Fixman has also given examples of unitary operators in $C(\Omega)$, the complex continuous functions on the compact Hausdorff space Ω, which are not spectral operators. In fact by demanding a countably additive resolution of the identity we probably rule out most of the differential operators in L_p with $p \neq 2$. However in many of the irregular cases of eigenvalue expansions associated with boundary value problems with discrete spectrum where the expansions are not unconditionally convergent (and thus the operators are not spectral operators) the general theory of spectral operators shows exactly how the expansion may be summed to the function being expanded.

A word about the organization of the present communication. In §2 the properties of spectral operators, their multiplicity theory, and the theory of algebras of spectral operators will be very briefly described. Practically no proofs will be given here as they are either to be found in the literature[7] or else they will appear in papers by Bade [4] or Foguel [17; 18; 19]. In §§3 and 4 however, we present in some detail, giving proofs and complete statements, the converse problem i.e., the problem of determining conditions on an operator sufficient to make it a spectral operator. In §3 four properties of the analytic functions $(\xi I - T)^{-1}x$ and their singularities are stated which are necessary and sufficient in order that an operator in a weakly complete space be a spectral operator. Because of their generality these conditions are quite difficult to apply in most concrete cases and for this reason we discuss, in §4, operators whose spectra lie in Jordan curves and whose resolvents have a finite rate of growth along a set of transversals to the Jordan curves. In this situation all but one of the four conditions required in §3 may be proved in abstract form and thus there is but one (the boundedness condition (B) of §3 which corresponds to (iii) above) to be verified in specific cases. The restrictions on the topological nature of the spectrum and on the rate of growth of the resolvent which are imposed in §4 are justified by the fact that they are satisfied by most operators arising from differential boundary value problems.

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[6] Communicated to the author by letter.

[7] Most of the results mentioned in §2 will be found in the references [1; 2; 3; 13; 16; 23; 37].

In fact the nth order singular differential operators with periodic coefficients have been shown by McGarvey to have their spectra lying in a finite set of analytic arcs. Also G. C. Rota has recently shown that the nth order singular operators whose coefficients are rational functions have their essential spectra in a finite set of analytic arcs. The resolvent in these cases has first order rate of growth along the normals to interior points of these analytic arcs. In these situations one finds examples of spectral operators which, unlike the situations considered by Neumark, are not similar to selfadjoint operators. For example McGarvey has shown that the second order operator $-(d/dt)^2+a(d/dt)+q(t)$ where a is a real constant not zero and q is a complex valued periodic function small relative to a determines a spectral operator in $L_2(-\infty, \infty)$. In this situation the spectrum need not be real and the operator need not be similar to a selfadjoint operator. If $a=0$ this is no longer true for McGarvey shows that the operator $-(d/dt)^2+\epsilon e^{2\pi i t}$ is not spectral for any $\epsilon\neq 0$. What destroys the spectral property in those cases which have been analyzed is the existence of a finite set of irregular branch points about which the spectral resolution is not countably additive.

A number of examples such as those studied by McGarvey, Neumark, and Rota, has led Schwartz to conjecture that the spectrum of a singular differential operator under quite general restrictions on the rate of growth of its coefficients and quite general boundary conditions will consist of a finite or enumerable number of analytic arcs running into and out of a finite or enumerable set of branch points, together with an enumerable, or vacuous, set of point eigenvalues whose only limit points are the branch points. It is further conjectured that the resolvent will have first order rate or growth along the normals to points interior to the analytic arcs and that the corresponding operators will be spectral operators in the complement of that portion of L_2 which is associated with an arbitrarily small neighborhood of the branch points.

Thus the theory in §§3 and 4 is presented in all detail in the hopes that it will interest others in the problem of verifying the *Schwartz conjecture.*

2. **Properties of spectral operators.** Before seeking conditions on the resolvent $R(\xi; T)$ of a linear operator T in a complex B-space $\mathfrak{X}$ which are sufficient to insure that T is a spectral operator we shall discuss here some of the properties of spectral operators and in particular see that the resolvent of such an operator does have properties not enjoyed by all resolvents. In other words, before seeking conditions sufficient for a spectral reduction we shall seek necessary conditions. Many of the results discussed are to be found in the literature or else will appear shortly in papers by S. R. Foguel and so proofs will be omitted. However a proof will be given for the canonical reduction of a spectral operator as it is more transparent than the one in the literature.[8] Recently a similar proof was communicated to me by Ciprian Foias.

Throughout this section T will be assumed to be a spectral operator in the complex B-space $\mathfrak{X}$. There are three properties of the resolvent $R(\xi; T)=(\xi I-T)^{-1}$ of the spectral operator T which are *not* properties of all resolvent operators but which are the basic properties of spectral operators that I wish to emphasize. In the following section it will be seen that these three properties come near to being sufficient for a spectral reduction and that for the operations arising in many mathematical problems the conditions (A) and (C) are automatically satisfied and only the boundedness condition (B) remains to be verified. The first of these properties is the *single valued extension property.*

(A) *For each x in X the function $R(\xi; T)x$ has the single valued extension property.*

To interpret this statement a vector valued function f is called an extension of $R(\xi; T)x$ if it is defined and analytic on an open set $D(f)$ containing the resolvent set $\rho(T)$ of T and if for each ξ in $D(f)$ we have $(\xi I-T)f(\xi)=x$. The function $R(\xi; T)x$ is said to have the single valued extension property if every pair f, g of extensions of $R(\xi; T)x$ have $f(\xi)=g(\xi)$ for ξ in $D(f)\cap D(g)$. The union of all the open set $D(f)$ as f varies over all extensions of $R(\xi; T)x$ is called the *resolvent set of* x and is denoted by the symbol $\rho(x)$. The *spectrum* $\sigma(x)$ of x is defined as the complement in the complex plane of the resolvent set of x. In view of (A) it is clear that there is a uniquely defined *maximal extension* $x(\xi)$, $\xi\in\rho(x)$ of $R(\xi; T)x$.

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[8] Theorem 8 in [13].

The second basic property of spectral operators is the *boundedness condition* expressed in the statement (B) that follows.

(B) *There is a constant K depending only upon T, such that for every pair x, y of vectors with $\sigma(x)$, $\sigma(y)$ disjoint we have*

$$|x| \leqq K|x+y|.$$

The third and final condition that is to be stressed is the following closure property.

(C) *For every closed set δ of complex numbers the set of all vectors x with $\sigma(x)\subseteq\delta$ is also closed.*

The set of vectors x whose spectrum lies in a given closed set δ is not only closed but it is precisely the range of the projection $E(\delta)$ associated with δ by the spectral resolution of T. It follows as a corollary that every bounded operator which commutes with T also commutes with every spectral resolution for T and this shows that the spectral resolution is uniquely determined by T. It should also be mentioned that $E(\sigma(T))=I$ and that $\sigma(x)$ is void if and only if $x=0$.

There is a canonical reduction for spectral operators which corresponds to the Jordan reduction for finite matrices of complex numbers. To explain this reduction we introduce the concept of a *scalar type operator* as an operator S which has a spectral resolution E for which

$$S = \int \lambda E(d\lambda).$$

This canonical reduction asserts that an operator T is a spectral operator if and only if it is the sum $T=S+N$ of a scalar type operator S and a quasi-nilpotent operator N commuting with S. Furthermore this decomposition is unique and T and S have the same spectrum and the same spectral resolution. The operator S is called the scalar part of T and N the radical part of T.

Here we have used the term quasi-nilpotent for an operator N for which $|N^n|^{1/n}\rightarrow 0$. It is clear that this condition is equivalent to the statement that the Laurent expansion for the resolvent $\Sigma N^n/\lambda^{n+1} =R(\lambda; N)$ is convergent for every $\lambda\neq 0$. Thus the operator N is quasi-nilpotent if and only if its spectrum $\sigma(N)=\{0\}$. S. R. Foguel [17] has shown that the scalar part S of a spectral operator T inherits many of the properties of T but it need not inherit the fine structure properties of the spectrum. One of Foguel's principal results asserts that if $\mathfrak{R}$ is *a uniformly closed right (or left) ideal in the algebra of bounded operators on $\mathfrak{X}$, and if T belongs to $\mathfrak{R}$ then so do S, N,* *and every projection* $E(\delta)$ *for* which $0\notin\bar{\delta}$. A number of interesting corollaries are immediate, e.g.: if T is compact so are S, N, and $E(\delta)$, $0\notin\bar{\delta}$ compact; if T is weakly compact so are S, N, and $E(\delta)$, $0\notin\bar{\delta}$ weakly compact; the ranges of S, N, and $E(\delta)$, $0\notin\bar{\delta}$ are contained in the closure of the range of T—thus if T is separable so are S, N, and $E(\delta)$, $0\notin\bar{\delta}$ separable; a similar result, also proved by Foguel, although not an immediate corollary of the above principle, asserts that S has a closed range if T does. Among other corollaries we mention the following: if $AT=0$ then $AS=AN=AE(\delta)=0$, $0\notin\bar{\delta}$; if $Tx=0$ then $Nx=Sx=E(\delta)x=0$, $0\notin\bar{\delta}$; if for some bounded sequence $\{x_n\}$ the sequence $\{Tx_n\}$ is convergent then if $0\notin\bar{\delta}$ the sequences $\{Sx_n\}$, $\{Nx_n\}$, $\{E(\delta)x_n\}$ are also convergent.

Foguel has also shown that for a bounded operator A, we have $TA=0$ *if and only if* $A=E(\{0\})A$ *and* $NE(\{0\})A=0$. It follows that if $E(\{0\})=0$ then $TA=0$ or $AT=0$ only when $A=0$ and that $\overline{T\mathfrak{X}}=\mathfrak{X}$. The following elementary examples are used by Foguel to show that the converses to a number of the above mentioned results are false. The operator $T=S+N$ in l_1 is defined by placing $S=0$ and $N(\xi_1, \xi_2, \cdots)=(\xi_2, 0, \xi_4, 0, \cdots)$. Then S is compact but T is not even weakly compact. In the space $C_0(0, 1)$, of continuous functions vanishing at the origin, let $T=S+N$ with $S=0$ and $(Nf)(t)=\int_0^t f(s)ds$. Then S has a closed range but T does not and here the point spectrum of S is the continuous spectrum of T. If, in this last example, we take S to be I instead of 0 then the ranges of T and S are both closed but the range of N is not closed.

Returning for a moment to the canonical reduction itself we shall indicate here an elementary proof which is independent of Gelfand's theory of normed rings. It will first be shown that the sum $T=S+N$ of a scalar operator S and a quasi-nilpotent N commuting with S is a spectral operator having the same spectral resolution as S. Let E be the spectral resolution for S so that for each Borel set δ the projection $E(\delta)$ commutes with N and thus with T. Thus to show that E is also a spectral resolution for T it suffices to show that $\sigma(T|E(\delta)\mathfrak{X})\subseteq\bar{\delta}$ for every Borel set δ. But, since E is a spectral resolution for S this will follow if it is shown that $\sigma(T|E(\delta)\mathfrak{X})=\sigma(S|E(\delta)\mathfrak{X})$. This latter fact follows (by replacing $\mathfrak{X}$ by $E(\delta)\mathfrak{X}$) from the equation $\sigma(S+N)=\sigma(S)$ which will now be derived in an elementary fashion rather than by appealing to the ideal theory in normed rings. Since N is quasi-nilpotent it follows that $|N^k|=O(\epsilon^k)$ for every $\epsilon>0$ and thus for every λ in $\rho(S)$ the series $\sum_{k=0}^{\infty} N^k R(\lambda; S)^k$ converges in the uniform operator topology. Since

$$\left\{\sum_{k=0}^{\infty} N^k R(\lambda; S)^k\right\}\{I - NR(\lambda;S)\} = \{I - NR(\lambda;S)\} \sum_{k=0}^{\infty} N^k R(\lambda; S)^k$$

$$= \sum_{k=0}^{\infty} N^k R(\lambda; S)^k - \sum_{k=1}^{\infty} N^k R(\lambda; S)^k = I,$$

the series is $\{I-NR(\lambda; S)\}^{-1}$. This shows that

$$(\lambda I - S - N)^{-1} = R(\lambda; S)\{I - NR(\lambda; S)\}^{-1}$$

exists as an everywhere defined and bounded operator. Thus $\lambda \in \rho(S+N)$ which shows that $\sigma(S) \supseteq \sigma(S+N)$. It follows as a corollary that $\sigma(S+N) \supseteq \sigma(S+N-N) = \sigma(S)$ and so $\sigma(S+N) = \sigma(S)$. Thus T is a spectral operator having the same spectral resolution as S. Since $S = \int \lambda E(d\lambda)$ it follows that S is uniquely determined by T. Hence $N = T - S$ is also uniquely determined by T.

Finally it will be shown that every spectral operator T has the desired decomposition. The operators S and N are defined by the equations

$$S = \int \lambda E(d\lambda), \qquad N = T - S$$

where E is the resolution of the identity for T. It is clear that S is a scalar type operator with resolution of the identity E. Also, since T commutes with $E(\delta)$ it commutes with S and thus N commutes with S. The desired conclusion will therefore be established as soon as it is shown that N is a quasi-nilpotent. To prove this we will show that the spectrum $\sigma(N)$ of N is contained in the circle $C_\epsilon = \{\lambda \mid |\lambda| \leqq \epsilon\}$ whose radius ϵ is an arbitrary positive number. Now let the spectrum of T be decomposed into the union of the disjoint Borel sets $\sigma_1, \cdots, \sigma_k$ each having diameter less than a positive number $\alpha < \epsilon$ which will be specified presently. If λ is in the resolvent set of each of the restrictions $N_{\sigma_i} = N \mid E(\sigma_i)\mathfrak{X}$, and if $R_i = R(\lambda, N_{\sigma_i})$, then, putting $R = \sum_{i=1}^{k} R_i E(\sigma_i)$, we have

$$(\lambda I - N)R = \sum_{i=1}^{k} (\lambda I - N_{\sigma_i}) R_i E(\sigma_i) = \sum_{i=1}^{k} E(\sigma_i) = I$$

and

$$R(\lambda I - N) = \sum_{i=1}^{k} R(\lambda I - N)E(\sigma_i) = \sum_{i=1}^{k} R(\lambda I - N_{\sigma_i})E(\sigma_i)$$

$$= \sum_{i=1}^{k} R_i(\lambda I - N_{\sigma_i})E(\sigma_i) = \sum_{i=1}^{k} E(\sigma_i) = I.$$

Thus λ is in $\rho(N)$. Consequently, the spectrum of N is contained in the union of the spectra $\sigma(N_{\sigma_i})$ of the restrictions $N|E(\sigma_i)\mathfrak{X}$, so that it will suffice to show that $\sigma(N_{\sigma_i})\subseteq C_\epsilon$ for each $i=1, \cdots, k$. To show this we write

$$N_{\sigma_i} = (T - \lambda_i I)_{\sigma_i} + (\lambda_i I - S)_{\sigma_i}$$

where λ_i is a point in $\sigma(T_{\sigma_i})$. Since $\sigma(T_{\sigma_i})\subseteq\bar{\sigma}_i$ we have,

$$\sigma((T - \lambda_i I)_{\sigma_i}) \subseteq \bar{\sigma}_i - \lambda_i \subseteq C_\alpha \subseteq C_\epsilon.$$

Since $(\lambda_i I - S)_{\sigma_i}$ is the restriction of $\int_{\sigma_i}(\lambda_i - \lambda)E(d\lambda)$ to σ_i we have

$$\left|(\lambda_i I - S)_{\sigma_i}\right| \leqq v(E) \max_{\lambda\in\sigma_i} \left|\lambda - \lambda_i\right| \leqq \alpha v(E)$$

and thus $(\lambda_i I - S)_{\sigma_i}$ is small in norm if α is small. Thus $\sigma(N_{\sigma_i})\subseteq C_\epsilon$ for small α. By the above, this shows that $\sigma(N)\subseteq C_\epsilon$ and since $\epsilon>0$ is arbitrary it follows that $\sigma(N)=\{0\}$. It then follows that N is a quasi-nilpotent. Q.E.D.

For functions f in the algebra $\mathfrak{F}(T)$ of all complex functions defined, single valued, and analytic on an open set containing the spectrum $\sigma(T)$ of T the formula

$$\text{(i)} \qquad f(T) = \frac{1}{2\pi i}\int_C f(\lambda)R(\lambda; T)d\lambda,$$

where C is a contour surrounding the spectrum of T, establishes a homomorphic map of $\mathfrak{F}(T)$ into the algebra of operators on $\mathfrak{X}$ which maps 1 into I and λ into T. Thus it establishes an operational calculus. For a spectral operator T with radical part N the formula (i) may be written as

$$\text{(ii)} \qquad f(T) = \sum_{n=0}^{\infty} \frac{N^n}{n!}\int f^{(n)}(\lambda)E(d\lambda),$$

the series being convergent in the uniform operator topology. Or, since $f^{(n)}(S)=\int f^{(n)}(\lambda)E(d\lambda)$, it may be written as

$$\text{(iii)} \qquad f(S + N) = \sum_{n=0}^{\infty} \frac{N^n}{n!} f^{(n)}(S),$$

a form which, as Schwartz has shown, [35] is applicable to any pair of commuting operators S and N provided that f is single valued and analytic on an open set containing the spectra $\sigma(S)$ and $\sigma(S+N)$. If, for each f in $\mathfrak{F}(T)$, the formula (ii) reduces to

$$\text{(iv)} \qquad f(T) = \sum_{m=0}^{n} \frac{N^m}{m!}\int f^{(m)}(\lambda)E(d\lambda),$$

the operator T is said to be of *type n*. The operator is of type n if and only if $N^{n+1}=0$ and it is a scalar operator if and only if it is of type zero. Foguel has shown that spectral operators of finite type can have no residual spectrum and that numbers λ in the point spectrum are characterized by the condition $E(\{\lambda\})\neq 0$. In general a spectral operator T may have a residual spectrum and the relations

$$\sigma_c(S) \subseteq \sigma_c(T), \qquad \sigma_p(T) \cup \sigma_r(T) \subseteq \sigma_p(S)$$

between the point, continuous, and residual spectra of T and its scalar part S have been established by Foguel.

J. Wermer [37] has shown that the scalar operators in Hilbert space are those operators which are similar to normal operators. More precisely, Wermer, using an idea of Mackey [26], which is based upon a powerful inequality of Lorch [25], has shown that for every finite set $S_1, \cdots, S_k$ of commuting scalar operators in Hilbert space there is a bounded selfadjoint operator B with a bounded everywhere defined inverse such that the operators BS_iB^{-1}, $i=1, \cdots, k$, are all normal. This shows that the sum and product of two commuting bounded spectral operators in Hilbert space are also spectral operators. As Kakutani [23] has shown this is not true in all B-spaces. In this connection however, it has been observed by Foguel [16] that in any space the sum (product) of two commuting spectral operators is a spectral operator if and only if the sum (product) of their scalar parts is a scalar type operator.

For applications to differential boundary value problems some of the properties of unbounded spectral operators must be developed. Most of the necessary work in this direction has been done by W. G. Bade [1]. We shall discuss here very briefly a few of the facts concerning unbounded operators. All proofs will be found either in Bade's work or in L.O. II.

While the theory of unbounded spectral operators is developed in the works cited *ab initio* and not made to depend in a fundamental way on the theory of bounded spectral operators there is a fundamental connection between bounded and unbounded spectral operators which is helpful to keep in mind. If the complex number λ is in the resolvent set of a closed operator T then T is a spectral operator if and only if its resolvent $R(\lambda; T)$ is a spectral operator whose spectral resolution E_1 has $E_1(\{0\})=0$. Another useful property relating the concepts of bounded and unbounded spectral operators is the following. If E is the spectral resolution for a closed spectral operator T then the restriction of T to $E(\delta)\mathfrak{X}$ is a spectral operator whose spectral resolution is the restriction of E to $E(\delta)\mathfrak{X}$ and, in particular, if δ is bounded then the restriction of T to $E(\delta)\mathfrak{X}$ is a bounded spectral operator. There is a commutativity property for unbounded closed spectral operators which is quite analogous in statement and proof to the one stated for bounded operators. Let T be such an operator with spectral resolution E and let A be a bounded operator with $A\mathfrak{D}(T)\subseteq\mathfrak{D}(T)$ and $ATx=TAx$ for x in $\mathfrak{D}(T)$. Then A commutes with all the projections $E(\delta)$. Just as in the bounded case this property may be used to prove the uniqueness of the spectral resolution of a closed operator. Bade has given examples to show that the canonical reduction theorem is not valid for unbounded spectral operators. However, if S is a closed scalar type spectral operator with spectral resolution E and if N is a bounded operator commuting with all the projections $E(\delta)$, then $S+N$ is a spectral operator with spectral resolution E provided that the restriction of N to each of the subspaces $E(\delta)\mathfrak{X}$, with δ bounded, is quasi-nilpotent.

An operational calculus exists for closed spectral operators and may be developed as follows. Let E be the resolution of the identity for the closed spectral operator T and let f be single valued and analytic on an open set G with $E(G)=I$. Let $\{\delta_n\}$ be an arbitrary increasing sequence of bounded Borel sets with closures contained in G and such that $E(\bigcup_{n=1}^{\infty}\delta_n)=I$. Then the operator $f(T)$ is defined by the equations

$$\mathfrak{D}(f(T)) = \left\{x \,\middle|\, \lim_{n\to\infty} f(T \mid E(\delta_n)\mathfrak{X})E(\delta_n)x \text{ exists}\right\},$$

$$f(T)x = \lim_{n\to\infty} f(T \mid E(\delta_n)\mathfrak{X})E(\delta_n)x, \qquad x \in \mathfrak{D}(f(T)).$$

The operator $f(T)$ thus defined is a closed linear operator which is independent of the particular sequence $\{\delta_n\}$ of Borel sets used to define it. If T is a closed scalar type operator the operational calculus may be extended from analytic functions to functions f which are Borel measurable on the spectrum of T. For such an f let f_n be defined by the equations

$$f_n(\lambda) = f(\lambda), \qquad |f(\lambda)| \leqq n;$$
$$f_n(\lambda) = 0, \qquad |f(\lambda)| > n.$$

Then the operator $f(T)$ may be defined by the equations

$$\mathfrak{D}(f(T)) = \left\{x \,\middle|\, \lim_{n} \int f_n(\lambda)E(d\lambda)x \text{ exists}\right\},$$

$$f(T) = \lim_{n} \int f_n(\lambda)E(d\lambda)x, \qquad x \in \mathfrak{D}(f(T)).$$

The operator $f(T)$ thus defined is a closed operator with dense domain and, of course, coincides with the previously defined operator $f(T)$ in case f is analytic. The usual rules for an operational calculus may be readily verified.

If T is a closed scalar type spectral operator with spectral resolution E and if f is a Borel measurable function then the operator $f(T)$ is also a closed scalar type spectral operator whose spectral resolution E_1 is given by the formula $E_1(\delta)=E(f^{-1}(\delta))$. In general if T is an arbitrary closed spectral operator with spectral resolution E and if f is a single valued function, analytic on a domain G which, when taken together with a finite number of exceptional points p includes a neighborhood of $\sigma(T)$ and a neighborhood of the point at infinity, then $f(T)$ is a closed spectral operator provided that, at each exceptional point p, as well as at ∞, f has at most a pole, and also $E(\{p\})=0$ for each exceptional point p. In particular a polynomial function of a closed spectral operator is a closed spectral operator.

Bade [**3**; **4**] has developed, by means of some remarkably powerful arguments, a theory of algebras of spectral operators which extends many features of the theory of commutative W^*-algebras in Hilbert space. In particular he has given answers to the following questions. If τ is a family of commuting scalar type spectral operators, when are all the operators in the weakly closed algebra generated by τ also scalar type spectral operators, and precisely what are the operators in this weakly closed algebra generated by τ? Before surveying this work of Bade on algebras of scalar type spectral operators I should like to describe briefly the structure of uniformly closed commutative algebras of general spectral operators. It will be convenient to use the term *full algebra* for a uniformly closed algebra of operators which contains the inverse of each of its nonsingular elements. The intersection of all the full algebras containing a given family of operators is called the full algebra generated by the family. A fundamental result in the study of algebras of spectral operators states that the uniformly closed algebra of operators generated by a bounded Boolean algebra of projections is a full algebra equivalent to the algebra of continuous functions on its own space of maximal ideals. From this it follows that if $\mathfrak{A}(\tau)$ is the full algebra generated by a family τ of commuting spectral operators in a B-space, $\mathfrak{X}$ together with their resolutions of the identity and if the Boolean algebra β generated by the spectral measures of the operators in τ is bounded then $\mathfrak{A}(\tau)$ is a vector direct sum

$$\mathfrak{A}(\tau) = \mathfrak{A}(\beta) + \mathfrak{R}$$

where $\mathfrak{R}$ is the radical in $\mathfrak{A}(\tau)$ and where $\mathfrak{A}(\beta)$ is the uniformly closed algebra generated by β. Furthermore $\mathfrak{A}(\beta)$ is equivalent to the algebra of continuous functions on the structure space Λ of $\mathfrak{A}(\tau)$. This latter isomorphism may be represented by an integral of the form

$$S^*(f) = \int_\Lambda f(\lambda) A(d\lambda), \qquad f \in C(\Lambda),$$

where A is a spectral measure in $\mathfrak{X}^*$ and where $S^*(f)$ is the adjoint of the operator $S(f)$ in $\mathfrak{A}(\beta)$ corresponding to f. If the space $\mathfrak{X}$ is weakly complete then A is the adjoint of a countably additive spectral measure E in $\mathfrak{X}$ and the preceding formula takes the form

$$S(f) = \int_\Lambda f(\lambda) E(d\lambda), \qquad f \in C(\Lambda).$$

This type of formula raises the natural question of considering the operators it defines for bounded Borel functions f which are not necessarily continuous. In this connection let us consider a countably additive spectral measure E on a σ-field Σ of subsets of a set Λ. Then a function f on Λ is said to be *E-essentially bounded on Λ* if

$$E - \operatorname*{ess\,sup}_{\lambda\in\Lambda} |f(\lambda)| = \inf_{E(\delta)=I} \sup_{\lambda\in\delta} |f(\lambda)|$$

is finite. Since E is countably additive on Σ there is a set δ_0 in Σ with $E(\delta_0)=I$ and for which

$$E - \operatorname*{ess\,sup}_{\lambda\in\Lambda} |f(\lambda)| = \sup_{\lambda\in\delta_0} |f(\lambda)|.$$

The map $f\rightarrow S(f)$ defined by the integral $S(f)=\int_\Lambda f(\lambda)E(d\lambda)$ is an *isomorphism* between the algebra $EB(\Lambda,\Sigma)$ of all E-essentially bounded Σ-measurable scalar functions on and a full algebra of scalar type spectral operators. The resolution of the identity $E(S(f))$ for $S(f)$ is given by the formula

$$E(S(f), \sigma) = E(f^{-1}(\sigma))$$

where σ is an arbitrary Borel set in the plane. The norm of $S(f)$ is related to that of f by the inequalities

$$E - \operatorname*{ess\,sup}_{\lambda\in\Lambda} |f(\lambda)| \leqq |S(f)| \leqq KE - \operatorname*{ess\,sup}_{\lambda\in\Lambda} |f(\lambda)|,$$

where K is independent of f. It follows that $S(f)$ has a bounded inverse if and only if f^{-1} is E-essentially bounded and that the spectrum of $S(f)$ is given by the formula

$$\sigma(S(f)) = \bigcap_{E(\delta)=I} \overline{f(\delta)}, \qquad f \in EB(\Lambda, \Sigma).$$

Also if the sequence $\{f_n\}$ is bounded in $EB(\Lambda, \Sigma)$ and if $f_n(\lambda) \rightarrow f(\lambda)$ except for those λ in a set of E-measure zero then $S(f_n)x \rightarrow S(f)x$ for every x in $\mathfrak{X}$. Another corollary is that an algebra of operators in a weakly complete space which is topologically and algebraically equivalent to some algebra of bounded continuous functions consists entirely of scalar type operators. Thus every operator in the uniformly closed algebra generated by a bounded Boolean algebra of projections in a weakly complete space is a scalar type spectral operator. It follows that if the Boolean algebra generated by the spectral measures of the operators in a commutative family τ of spectral operators is bounded then every operator in the full algebra $\mathfrak{A}(\tau)$ generated by τ is a spectral operator. In Hilbert space the Boolean algebra generated by the spectral measures of a finite set $T_1, \cdots, T_n$ of commuting spectral operators is always bounded and so the full algebra $\mathfrak{A}$ generated by $T_1, \cdots, T_n$ and their spectral measures has the form $\mathfrak{A} = \mathfrak{B} + \mathfrak{R}$ where $\mathfrak{R}$ is the radical in $\mathfrak{A}$ and where $\mathfrak{B}$ consists of scalar type spectral operators and is the algebra generated by the spectral measures for $T_1, \cdots, T_n$. If $n=1$ this result is true for any B-space $\mathfrak{X}$ and furthermore every operator in $\mathfrak{B}$ is an E-essentially bounded function of the scalar part of T_1 (E is the spectral measure for T_1).

The preceding results concerning uniformly closed algebras are not difficult to prove and, for the most part may be found in [**13**]. We shall now describe Bade's results on strongly or weakly closed algebras of spectral operators. Since a convex set in the space of all bounded linear maps between two B-spaces has the same closure in the weak as in the strong operator topology *the strong and weak operator closures of an algebra of operators are the same.* Bade's basic result is the following theorem.

THEOREM (BADE). *Let β be a bounded Boolean algebra of projections in a weakly complete B-space $\mathfrak{X}$. Then the weakly closed operator algebra generated by β consists of all operators in $\mathfrak{X}$ which leave invariant every closed linear manifold which is left invariant by every member of β.*

One of the tools developed by Bade in his study is the following lemma which enables him to dispense with the scalar product in Hilbert space and thus work in more general B-spaces. The lemma concerns a σ-complete Boolean algebra $\mathfrak{B}$ of projections in a B-space $\mathfrak{X}$, and asserts that for each x_0 in $\mathfrak{X}$ there is a linear functional x_0^* in $\mathfrak{X}^*$ with the properties that $x_0^* E x_0 \geqq 0$ for E in $\mathfrak{B}$ and also that if

$x_0^*Ex_0=0$ for some E in $\mathfrak{B}$ then $Ex_0=0$. A corollary of Bade's theorem is that every operator in the weakly closed algebra of operators in a weakly complete space which is generated by a bounded Boolean algebra B of projections is a scalar type spectral operator. If in addition it is known that for some vector x the set $\{Ex \mid E\in\mathfrak{B}\}$ is a fundamental in $\mathfrak{X}$ then a bounded linear operator is in the weakly closed algebra generated by $\mathfrak{B}$ if and only if it commutes with every element of $\mathfrak{B}$. Another corollary is that every operator in the weakly closed operator algebra generated by a spectral operator of scalar type and its spectral measure is a spectral operator of scalar type.

Another result of Bade, which belongs to that interesting collection of principles dealing with conditions under which weak convergence implies strong convergence, is the following theorem.

If a generalized sequence of projections in a σ-complete Boolean algebra of projections converges weakly to a projection then it converges strongly. If the space is weakly complete then the assumption of σ-completeness may be replaced by that of boundedness. It has also been shown by Bade that *if the Boolean algebra is complete* then *the weakly closed operator algebra it generates is the same as the uniformly closed algebra it generates.*

Using some of the results on algebras of spectral operators Foguel [19] has proved the following perturbation theorem. *Let $\{S_n\}$ be a sequence of commuting scalar type operators which converges strongly to the operator S. Let the Boolean algebra generated by the spectral measures of the operators S_n be bounded. Then S is a scalar type operator whose spectral measure E is related to the spectral measure E_n of S_n by the equation $E(\sigma)x=\lim_n E_n(\sigma)$ for every x in $\mathfrak{X}$ and Borel set σ for which* E (boundary $\sigma)x=0$.

One feature of the theory of algebras of spectral operators which has, until recently, been neglected is the multiplicity theory. In 1956 Dieudonné [6] obtained such a theory under the assumption that the adjoint $\mathfrak{X}^*$ of the underlying space is separable. More recently Bade [4] has developed a new approach to multiplicity theory which does not require the separability assumption. We shall describe briefly Bade's theory. A cardinal valued function m on a *complete* abstract Boolean algebra $\mathfrak{B}$ is called a *multiplicity function* if $m(0)=0$ and if $m(\vee E_\alpha)=\vee m(E_\alpha)$ for every set $\{E_\alpha\}$. The cardinal number $m(E)$ is called the multiplicity of E. The element E is said to have *uniform multiplicity* n if $m(F)=n$ whenever $0\neq F\leqq E$. Any such multiplicity function m on $\mathfrak{B}$ uniquely determines a family $\{E_n\}$, $n\leqq m(I)$, of disjoint elements of $\mathfrak{B}$ such that $I=\vee_n E_n$ and E_n, if not 0, has uniform multiplicity n. What is desired in case the Boolean algebra $\mathfrak{B}$ is a Boolean algebra of projections in the B-space $\mathfrak{X}$ is a multiplicity function which is a *natural* one in the sense that $m(E)$ is the least number of cyclic subspaces spanning the range of E. (The cyclic subspace $\mathfrak{M}(x)$ spanned by a vector x is the closed linear manifold determined by all vectors of the form Ex, $E \in \mathfrak{B}$.) If every family of disjoint projections in $\mathfrak{B}$ which is bounded by a given projection E is at most countable then E is said to satisfy the countable chain condition. Such E form a dense σ-ideal in $\mathfrak{B}$ and what Bade does is to define $m(E)$ for such E as the least cardinal number of cyclic subspaces spanning the range of E and then prove that m has a unique extension to a multiplicity function defined on all of $\mathfrak{B}$. Suppose that the identity I in $\mathfrak{B}$ has a finite uniform multiplicity n and that $\mathfrak{X} = \bigvee_{i=1}^{n} \mathfrak{M}(x_i)$. Let us regard $\mathfrak{B}$ as a spectral measure on the Borel sets Σ of its Stone space Ω. Bade shows that there are functionals x_i^*, $i = 1, \cdots, n$, with $x_i^* \mathfrak{M}(x_j) = 0$, $j \neq i$, and that the measure $x_i^* E(\sigma) x_i = \mu_i(\sigma)$ is positive and dominates the vector measure $E(\sigma)x_i$ and uses these measures to define the direct sum $L = \sum_{i=1}^{n} L_1(\Omega, \Sigma, \mu_i)$. The representation of $\mathfrak{X}$ and $\mathfrak{B}$ may now be stated as follows. There is a linear continuous one to one map T of $\mathfrak{X}$ into a dense subspace of the direct sum L such that if $Tx = [f_1, \cdots, f_n]$ then

$$x_i^* E(e) x_i = \int_e f_i(\omega) \mu_i(d\omega), \qquad e \in \Sigma, i = 1, \cdots, n,$$

$$x = \lim_{m \to \infty} \sum_{i=1}^{n} \int_{e_m} f_i(\omega) E(d\omega) x_i,$$

where $e_m = \{\omega \mid |f_i(\omega)| \leqq m, i = 1, \cdots, n\}$. This is not as complete a description as one has for the well known Hilbert space case but, of course, it is not to be expected that T maps $\mathfrak{X}$ onto all of L.

Bade proves that a projection E in $\mathfrak{B}$ has finite uniform multiplicity n if and only if its adjoint E^* has finite uniform multiplicity n. As a corollary he obtains the important result that if $\mathfrak{X}$ is separable then $m(E) = m(E^*)$ for every E in $\mathfrak{B}$. The main question left unanswered by Bade's work is the relation between $m(E)$ and $m(E^*)$ for projections of infinite multiplicity on nonseparable spaces.

Using the spectral representation associated with the spectral multiplicity theory of normal operators in Hilbert space and a result of Bade's, Foguel [18] has recently obtained some interesting results concerning the representation of operators in a separable Hilbert space $\mathfrak{H}$ which commute with a given bounded normal operator S of finite multiplicity n in $\mathfrak{H}$. In view of the classical spectral representation theory there is a finite decreasing sequence $e_1 \supset e_2 \supset \cdots \supset e_n$, of

Borel sets in the complex plane with, e_1 the whole plane, and Borel measures μ_j with $\mu_j(e)=\mu(ee_j)$, $j=1, \cdots, n$, such that $\mathfrak{H}$ is equivalent to the direct sum $\mathfrak{H}_1=\sum_{j=1}^{n} \oplus L_2(\mu_j)$. Under this equivalence the operator S becomes the operator S_1 of multiplication, i.e., the operator which maps the vector $\{f_1(\lambda), \cdots, f_n(\lambda)\}$ in $\mathfrak{H}_1$ into the vector $\{\lambda f_1(\lambda), \cdots, \lambda f_n(\lambda)\}$. Foguel's results are most clearly stated in terms of S_1 and $\mathfrak{H}_1$ and so we shall assume that S is the operation of multiplication by the independent variable in the Hilbert space $\mathfrak{H} = \sum_{i=1}^{n} \oplus L_2(\mu_i)$. Let $\mathfrak{A}$ be the algebra of bounded operators in $\mathfrak{H}$ which commute with S and $\mathfrak{A}_1$ the algebra of $n\times n$ matrices $A(\lambda) = (a_{ij}(\lambda))$ of bounded measurable functions a_{ij} for which $a_{ij}(\lambda)=0$ outside the set $e_i\cap e_j$. Then to each operator $A\in\mathfrak{A}$ corresponds a matrix $A(\lambda)$ which represents it in the sense that A transforms the vector $[f_1, \cdots, f_n]$ into the vector $[\sum a_{ij}(\lambda)f_j(\lambda), \cdots, \sum a_{nj}(\lambda)f_j(\lambda)]$. This representation is unique (up to sets of measure zero) and the mapping $A\rightarrow A(\lambda)$ is a $*$-homomorphism. If $\mathfrak{A}_1$ is normed by the expression $\max_{i,j}$ ess $\sup_\lambda |a_{ij}(\lambda)|$ then the mapping $A\rightarrow A(\lambda)$ is a homeomorphism. Strong convergence in $\mathfrak{A}$ is equivalent to uniform boundedness and convergence in measure (of the elements $a_{ij}(\lambda)$) in $\mathfrak{A}_1$. It follows as a corollary that if a sequence $\{A_n\}$ in $\mathfrak{A}$ converges strongly to the operator A then A_n^* converges strongly to A^*.

Foguel has also given an interesting canonical reduction for the matrices $A(\lambda)$. There are n bounded measurable functions $z_1(\lambda), \cdots, z_n(\lambda)$, and n disjoint projections $\epsilon_1(\lambda), \cdots, \epsilon_n(\lambda)$ whose elements are measurable and whose sum is the unit matrix, and a matrix $N(\lambda)$ which commutes with $\epsilon_i(\lambda)$, has measurable components, is nilpotent of order n, and for which

$$A(\lambda) = \sum_{i=1}^{n} z_i(\lambda)\epsilon_i(\lambda) + N(\lambda).$$

There is an increasing sequence $\{\alpha_m\}$ whose union is the complex plane and such that A restricted to $E(\alpha_m)\mathfrak{H}$ (E is the resolution of the identity for S) is a spectral operator. Thus every operator A in $\mathfrak{A}$ is the strong limit of a sequence of spectral operators. The operator A will itself be a spectral operator if and only if the elements of the matrices $\epsilon_1, \cdots, \epsilon_n$ are almost everywhere bounded. It also follows that any generalized nilpotent in $\mathfrak{A}$ must be a nilpotent of order n.

The operator A is compact if and only if there is a sequence $\{\lambda_n\}$ of eigenvalues for S for which $A(\lambda)=0$ for almost all λ not in the sequence $\{\lambda_n\}$ and $\lim_n A(\lambda_n)=0$. Consequently if S has no eigenvalues then $A=0$ and if S has only a finite number of eigenvalues then A has a finite dimensional range. If $\lambda\notin\{\lambda_n\}$ then $z_i(\lambda)=0$, $N(\lambda)=0$,

$\epsilon_1(\lambda)=I$, and $\epsilon_i(\lambda)=0$, $i\geqq 2$.

Since every scalar type spectral operator in Hilbert space is equivalent to a normal operator the above results hold for the algebra A of bounded operators which commute with a given bounded scalar type operator S of finite multiplicity.

3. **Conditions on the resolvent of an operator which are sufficient to insure that it is a spectral operator.** In the preceding section it was observed that the resolvent of a spectral operator has the special properties (A), (B), (C) listed there, which are not properties of resolvents in general. In this section and the next it will be shown that these conditions and, in general, another condition on the space $\mathfrak{X}$ or the operator T will prove to be sufficient to insure that T is a spectral operator. In order to simplify the discussion we shall restrict ourselves to bounded operators. The central ideas to be developed in detail in §§3 and 4 are in the literature[9] but the treatment here is more general and at the same time more transparent since a number of unnecessary concepts have been eliminated from the earlier development and new proofs have replaced some of the original ones. This section will be divided into three parts. Part 3(A), will discuss consequences of the condition (A), Part 3(B), those of conditions (A) and (B), while Part 3(C) will discuss consequences of the conditions (A), (B), and (C). In these three parts it will be shown that an operator T (in a weakly complete space) which has the properties (A), (B), and (C) has a uniquely determined countably additive spectral resolution defined on a σ-field $M(T)$ of sets measurable-T. In general this field need not contain all Borel sets and may not even contain enough sets to be useful at all. In Part 3(C) a fourth condition (D) is introduced so that the sets (A), $\cdots$, (D) are necessary as well as sufficient conditions for the operator T to be a spectral operator.

The conditions (A), $\cdots$, (D) are in abstract form and difficult to verify in most concrete problems. In order to make the analytical work of verification easier we will, in §4, introduce a class of operators whose resolvents obey a growth condition (G). This class is general enough to include most differential operators and restrictive enough so that the conditions (A), (C), and (D) may be verified. Thus, for a large class of operators the two conditions (B) and (G) may replace the four conditions (A), · · · , (D). For this reason the problem of verifying the boundedness condition (B) must be regarded as a key problem in applying the theory of spectral operators. The reader will readily perceive that (B) is not the sort of condition that is likely to be easy to verify. This is not surprising, for what (B) amounts to in the final analysis is the assertion that the resolution of the identity is countably additive. Thus (B) is the condition which gives rise to the phenomenon of unconditional convergence of the eigenvalue expansions.

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[9] These ideas were developed rather explicitly in [12] under the assumption that the spectrum is nowhere dense although in this work the present development was indicated. During the years 1953–1954 I had some correspondence with Gerhard Neubauer who made valuable suggestions concerning the work in [11] and [12]. In particular Neubauer observed that the single valued extension property of the resolvent was a necessary condition for the operator to be spectral. He also suggested a number of improvements that could be made in the theory presented in [12]. A summary of these ideas may be found in [9; 10; 11]. Also J. T. Schwartz has been a valuable critic and has made numerous contributions which have been incorporated in the presentation given in §§3 and 4.

What we mean, in more detail, is this. It is because of the countable additivity of the resolution of the identity that spectral operators have important expansion theorems associated with them. For example, if T is a spectral operator and its spectrum is denumerable, then every x in $\mathfrak{X}$ has an unconditionally convergent expansion of the type $x=\sum x_n$ $(=\sum_{\lambda\in\sigma(T)} E(\lambda)x)$ where the spectrum of x_n consists of "generalized eigenvector" associated with λ_n. If T is a spectral operator of scalar type, then the generalized eigenvectors are simply eigenvectors in the ordinary sense. If T is a spectral operator of type m, then the generalized eigenvectors x_n satisfy the equations $(\lambda_n I-T)^{m+1}x_n=0$, $n=1, 2, \cdots$. Thus, as long as T has a countably additive resolution of the identity, we are not far from the situation characteristic of normal operators in Hilbert space. If the countable additivity of the spectral resolution fails, so do many other convenient eigenvalue expansion properties.

It should be noted that it is by no means the case that all the familiar eigenvalue expansions of classical analysis are unconditionally convergent. Indeed, there are many examples, such as Fourier series expansions in the space $L_p(0, 2\pi)$ with $1<p<\infty$, where the expansion converges, but only conditionally. This seems to indicate that further developments in spectral theory will include a theory of conditionally convergent expansions associated with discrete and continuous spectra. Contributions along these lines have been made by D. R. Smart [36]. Nevertheless, the cases where one does have unconditionally convergent eigenvalue expansions are of sufficient importance to justify studying them for their own sake. It is this fact that lends importance to the problem of discovering which operators are spectral operators.

3(A). **Consequences of the condition** (A). Here we shall be concerned with a bounded linear transformation T in a complex B-space $\mathfrak{X}$. We shall establish a few properties of T on the basis of the single assumption (A), which is repeated here for convenience of reference.

(A) *For each x in $\mathfrak{X}$ the function $R(\xi; T)x$ has the single valued extension property.*

Since T satisfies condition (A) we recall that the *resolvent set* of x, which is denoted by $\rho(x)$, may be defined as the union of all the domains $D(f)$, the union being taken as f varies over all analytic extensions of $R(\xi; T)x$. Thus $\rho(x)$ is an open set containing $\rho(T)$, and its complement $\sigma(x)$ is a closed subset of $\sigma(T)$. The set $\sigma(x)$ is called the *spectrum* of x. It is clear that there is a unique maximal analytic extension of $R(\xi; T)x$. This extension, which is a single valued analytic function defined on $\rho(x)$ will be denoted by $x(\cdot)$. Thus the function $x(\cdot)$ has, by definition, the properties

$$(\xi I - T)x(\xi) = x, \qquad \xi \in \rho(x),$$
$$x(\xi) = R(\xi; T)x, \qquad \xi \in \rho(T).$$

Even though (A) is taken as a standing assumption throughout Part 3(A), it will be indicated parenthetically in the statement of each lemma in the proof of which it is used.

Lemma 1. (A) *If α, β are complex numbers and x, y are vectors in $\mathfrak{X}$, then*

$$\sigma(x + y) \subseteq \sigma(x) \cup \sigma(y),$$
$$\alpha x(\xi) + \beta y(\xi) = (\alpha x + \beta y)(\xi), \qquad \xi \in \rho(x)\rho(y).$$

Proof. The function $\alpha x(\xi) + \beta y(\xi)$ is an analytic extension of

$$R(\xi; T)\alpha x + R(\xi; T)\beta y = R(\xi; T)(\alpha x + \beta y), \qquad \xi \in \rho(T),$$

defined on the open set $\rho(x)\rho(y)$. Thus $\rho(\alpha x+\beta y) \supset \rho(x)\rho(y)$. For $\xi \in \rho(x)\rho(y)$ we have,

$$(\alpha x + \beta y)(\xi) = \alpha x(\xi) + \beta y(\xi),$$

by (A). Q.E.D.

Lemma 2. (A) *The spectrum $\sigma(x)$ is void if and only if $x = 0$.*

Proof. If $\sigma(x)$ is void then $x(\xi)$ is an entire function. Since

$$\lim_{\xi \to \infty} x^* x(\xi) = \lim_{\xi \to \infty} x^* R(\xi : T)x = 0,$$

it is seen that $x^* x(\xi) = 0$ for all ξ and all x^* in the conjugate space $\mathfrak{X}^*$. Hence

$$x^*x = x^*(\xi I - T)x(\xi) = \{(\xi I - T)^*x^*\}(x(\xi)) = 0,$$

for all x^* in $\mathfrak{X}^*$, and thus it follows from the Hahn-Banach extension theorem that $x=0$. Q.E.D.

LEMMA 3. (A) *Let σ be a set of complex numbers, and σ' its complement. If $x+y=x_1+y_1$, where $\sigma(x)$, $\sigma(x_1)\subseteq\sigma$ and $\sigma(y)$, $\sigma(y_1)\subseteq\sigma'$, then $x=x_1$, $y=y_1$.*

PROOF. By Lemma 1

$$\sigma(x - x_1) \subseteq \sigma(x) \cup \sigma(x_1),$$

$$\sigma(y_1 - y) \subseteq \sigma(y) \cup \sigma(y_1)$$

and so the vector $x-x_1=y_1-y$ has a void spectrum. Thus Lemma 2 shows that $x=x_1$, $y=y_1$. Q.E.D.

LEMMA 4. (A) *If P is a bounded linear operator in $\mathfrak{X}$ which commutes with T, then*

$$\sigma(Px) \subseteq \sigma(x), \qquad x \in \mathfrak{X}.$$

PROOF. Since P commutes with T it commutes with the resolvent $R(\xi; T)$ for every ξ in $\rho(T)$. From the equation $R(\xi; T)Px=PR(\xi; T)x$ it is clear that $Px(\xi)$ is an analytic extension of $R(\xi; T)Px$ to the domain $\rho(x)$. Thus $\rho(Px)\supseteq\rho(x)$ and hence $\sigma(Px)\subseteq\sigma(x)$. Q.E.D.

3(B). **Consequences of the conditions (A) and (B).** Throughout this section it will be assumed that the operator T satisfies condition (A) of 3(A) as well as the fundamental boundedness condition (B), which we restate here for convenience.

(B) *There is a constant K, depending only upon T, such that for every pair x, y of vectors with $\sigma(x)$, $\sigma(y)$ disjoint we have*

$$|x| \leqq K|x+y|.$$

Although the assumptions (A) and (B) will be standing assumptions throughout Part 3(B), they will be indicated in the statement of each lemma where they are used.

Assumption (B) allows us to associate projections $E(\delta)$ with certain sets δ of complex numbers.

DEFINITION 1. The symbol $S_1(T)$ will be used for the family of all sets σ with the property that vectors of the form $x+y$ with $\sigma(x)\subseteq\sigma$, $\sigma(y)\subseteq\sigma'$ are dense in $\mathfrak{X}$.

It is clear that if σ is in $S_1(T)$, then the complement σ' is also in $S_1(T)$.

LEMMA 2. (A, B) *If* σ *is in* $S_1(T)$, *there is one and only one bounded projection* $E(\sigma)$ *on* $\mathfrak{X}$ *with the properties* $E(\sigma)x=x$ *if* $\sigma(x)\subseteq\sigma$ *and* $E(\sigma)x=0$ *if* $\sigma(x)\subseteq\sigma'$. *Moreover,*

$$E(\sigma) + E(\sigma') = I, \qquad E(\sigma)E(\sigma') = 0, \qquad |E(\sigma)| \leqq K.$$

PROOF. The properties

$$(*) \qquad E(\sigma)x = x \text{ if } \sigma(x) \subseteq \sigma, \qquad E(\sigma)x = 0 \text{ if } \sigma(x) \subseteq \sigma',$$

define the projection $E(\sigma)$ on the dense set $D=\{x+y\mid\sigma(x)\subseteq\sigma, \sigma(y)\subseteq\sigma'\}$. Thus the uniqueness of $E(\sigma)$ is assured by the requirement that it is bounded.

To prove that an $E(\sigma)$ with the properties (*) exists, note that by Lemma 3 of Part 3(A) the properties (*) define a single-valued projection on D. Assumption (B) merely states that this projection is bounded, with bound at most K. Thus it has a unique extension by continuity to a projection with bound at most K defined on $\mathfrak{X}$. Since it is clear that $(E(\sigma)+E(\sigma'))x=x$ and $E(\sigma)E(\sigma')x=0$ for x in D, it follows by continuity that these properties hold for all x in $\mathfrak{X}$. Q.E.D.

LEMMA 3. (A, B) *If* P *is a bounded linear operator which commutes with* T, *then*

$$PE(\sigma) = E(\sigma)P, \qquad \sigma \in S_1(T).$$

PROOF. For σ in S_1, vectors of the form $z=x+y$ with $\sigma(x)\subseteq\sigma$ and $\sigma(y)\subseteq\sigma'$ are dense in $\mathfrak{X}$. For such a vector z we have $Pz=Px+Py$ and, by Lemma 4 of Part 3(A), $\sigma(Px)\subseteq\sigma$ and $\sigma(Py)\subseteq\sigma'$. Thus, by Lemma 2, $E(\sigma)Px=Px$ and $E(\sigma)Py=0$ and so by Lemma 2

$$E(\sigma)Pz = Px = PE(\sigma)z.$$

Since the vectors z are dense in $\mathfrak{X}$ we have $E(\sigma)P=PE(\sigma)$. Q.E.D.

We now introduce a subclass of $S_1(T)$, which will be shown to be a Boolean algebra.

DEFINITION 4. (A, B) The symbol $S_2(T)$ will be used for the family of all sets σ having the property that for every x in $\mathfrak{X}$ and every $\epsilon>0$, there are vectors x_1, x_1' with $\sigma(x_1)\subseteq\sigma(x)\sigma$, $\sigma(x_1')\subseteq\sigma(x)\sigma'$ and $|x_1+x_1'-x|<\epsilon$.

It is clear that $S_2(T)$ is closed under complementation, and contains the void set and the whole plane.

LEMMA 5. (A, B) *The family* $S_2(T)$ *is a Boolean algebra.*

PROOF. Since $S_2(T)$ is closed under complementation, to prove the lemma it is sufficient to show that it contains the union of every pair of its elements.

Let σ_1 and σ_2 be sets in $S_2(T)$ and, for every set μ of complex numbers, let

$$\mathfrak{M}(\mu) = \{x \mid \sigma(x) \subseteq \mu\}$$

for every subset μ of the complex plane. If x is in $\mathfrak{X}$, then, since σ_1 is in $S_2(T)$, x is in the closure of $\mathfrak{M}(\sigma_1\sigma(x))+\mathfrak{M}(\sigma_1'\sigma(x))$. On the other hand, it follows, since σ_2 is in $S_2(T)$, that $\mathfrak{M}(\sigma_2\sigma_1'\sigma(x))+\mathfrak{M}(\sigma_2'\sigma_1'\sigma(x))$ is dense in $\mathfrak{M}(\sigma_1'\sigma(x))$. Since, by Lemma 1 of Part 3(A), $\mathfrak{M}(\sigma_1\sigma(x))$ $+\mathfrak{M}(\sigma_2\sigma_1'\sigma(x))$ is contained in $\mathfrak{M}((\sigma_1\cup\sigma_2)\sigma(x))$, it follows immediately that x is in the closure of $\mathfrak{M}((\sigma_1\cup\sigma_2)\sigma(x))+\mathfrak{M}((\sigma_1\cup\sigma_2)'\sigma(x))$. But this means that $\sigma_1\cup\sigma_2$ is in $S_2(T)$. Q.E.D.

LEMMA 6. (A, B) *The restriction of the projection valued function E from $S_1(T)$ to the Boolean algebra $S_2(T)$ is a spectral measure.*

PROOF. The term spectral measure is used for a homomorphic map from a Boolean algebra into an algebra of projections provided that the units of the Boolean algebra map into the projection operators 0 and I respectively. We use the notations of the proof of the preceding lemma and let σ be in $S_2(T)$. Since an arbitrary vector is in the closure of $\mathfrak{M}(\sigma\sigma(x))+\mathfrak{M}(\sigma'\sigma(x))$ and since, by Lemma 2, $E(\sigma)(z+y)=z$ for z in $\mathfrak{M}(\sigma\sigma(x))$ and y in $\mathfrak{M}(\sigma'\sigma(x))$, it follows that $E(\sigma)x$ is in Cl $\mathfrak{M}(\sigma\sigma(x))$ (where the symbol Cl $\mathfrak{M}(\sigma\sigma(x))$ is used for the closure of $\mathfrak{M}(\sigma\sigma(x))$, and that $E(\sigma)x=x$ for x in Cl $\mathfrak{M}(\sigma\sigma(x))$. Thus, if σ_1, σ_2 are in $S_2(T)$, then we have $E(\sigma_1)E(\sigma_2)x$ is in Cl $\mathfrak{M}(\sigma_1\sigma_2\sigma(x))$. Hence

$$E(\sigma_1\sigma_2)E(\sigma_1)E(\sigma_2)x = E(\sigma_1)E(\sigma_2)x.$$

Since

$$E(\sigma_1\sigma_2)x \in \mathrm{Cl}\,\mathfrak{M}(\sigma_1\sigma_2\sigma(x)) \subseteq \mathrm{Cl}\,\mathfrak{M}(\sigma_1\sigma_2(x)) \cap \mathrm{Cl}\,\mathfrak{M}(\sigma_2\sigma(x))$$

we have

$$E(\sigma_1)E(\sigma_2)E(\sigma_1\sigma_2)x = E(\sigma_1\sigma_2)x.$$

Since all the projections $E(\cdot)$ commute with T and hence with each other, by Lemma 3, it follows that $E(\sigma_1)E(\sigma_2)=E(\sigma_1\sigma_2)$. Then we have

$$\begin{aligned} E(\sigma_1) \vee E(\sigma_2) &= E(\sigma_1) + E(\sigma_2) - E(\sigma_1\sigma_2) \\ &= I - (I - E(\sigma_1))(I - E(\sigma_2)) \\ &= I - (E(\sigma_1')E(\sigma_2')) = I - E(\sigma_1'\sigma_2') \\ &= E((\sigma_1'\sigma_2')') = E(\sigma_1 \cup \sigma_2). \end{aligned}$$

Q.E.D.

DEFINITION 7. (A, B) The symbol $S(T)$ will be used for the collection of those sets σ in $S_2(T)$ for which there exist closed sets μ_n, ν_n in $S_2(T)$ with $\mu_n \subseteq \sigma$, $\nu_n \subseteq \sigma'$, $n=1, 2, \cdots$ and

$$x = \lim_{n\to\infty} [E(\nu_n) + E(\mu_n)]x, \qquad x \in \mathfrak{X}.$$

Lemma 8. (A, B) *The family $S(T)$ is a Boolean algebra.*

Proof. It is clear that $S(T)$ is closed under complementation. Hence, in order to show that $S(T)$ is a Boolean algebra, it will suffice to show that it is closed under the operation of forming unions.

Let σ, $\tilde{\sigma}$ be in $S(T)$. Let $\{\mu_n\}$ and $\{\nu_n\}$ be as in Definition 7, and let $\{\tilde{\mu}_n\}$ and $\{\tilde{\nu}_n\}$ be sequences of closed sets in $S_2(T)$ such that $\tilde{\mu}_n \subseteq \tilde{\sigma}$, $\nu_n \subseteq \tilde{\sigma}'$, and

$$x = \lim_{n\to\infty} \{E(\tilde{\nu}_n)x + E(\tilde{\mu}_n)x\}, \qquad x \in \mathfrak{X}.$$

Since the sequence $\{E(\nu_n)+E(\mu_n)\}$ is strongly convergent it is bounded and the operators $E(\nu_n)+E(\mu_n)$ are therefore equi-continuous. Thus we have

$$\begin{aligned} x &= \lim_{n\to\infty} [E(\nu_n) + E(\mu_n)][E(\tilde{\mu}_n) + E(\tilde{\nu}_n)]x \\ &= \lim_{n\to\infty} \{E(\mu_n\tilde{\mu}_n \cup \mu_n\tilde{\nu}_n \cup \nu_n\tilde{\mu}_n)x + E(\nu_n\tilde{\nu}_n)x\}, \qquad x \in \mathfrak{X}. \end{aligned}$$

Since $\{\mu_n\tilde{\mu}_n \cup \mu_n\tilde{\nu}_n \cup \nu_n\tilde{\mu}_n\}$ and $\{\nu_n\tilde{\nu}_n\}$ are sequences of closed sets in $S_2(T)$ contained in $\sigma\cup\tilde{\sigma}$ and $(\sigma\cup\tilde{\sigma})'$ respectively, it follows that $\sigma\cup\tilde{\sigma}$ is in $S(T)$. Q.E.D.

Lemma 9. (A, B) *The set $\sigma(T)$ belongs to $S(T)$, and $E(\sigma(T))=I$. Furthermore, every subset δ of the resolvent set $\rho(T)$ is in $S(T)$ and has $E(\delta)=0$.*

Proof. Since $\sigma(x)\subseteq\sigma(T)$ for all x in $\mathfrak{X}$, it is clear from Definition 4 and Lemma 2 that $\sigma(T)$ is in $S_2(T)$ and that $E(\sigma(T))x=x$ for x in $\mathfrak{X}$. Since $\sigma(T)$ is closed, it is clear from Definition 7 that $\sigma(T)$ is in $S(T)$. If $\delta\subseteq\rho(T)$ then the void set ϕ and the spectrum $\sigma(T)$ are closed subsets of δ, δ' respectively and $E(\phi)x+E(\sigma(T))x=x$ which proves that δ is in $S(T)$. Since $E(\rho(T))=0$ we have $E(\delta)=E(\delta)E(\rho(T))=0$. Q.E.D.

Lemma 10. (A, B) *Let $\{\sigma_m\}$ be a decreasing sequence of sets in $S(T)$ whose limit σ is also in $S(T)$. Then*

$$E(\sigma)x = \lim_{m\to\infty} E(\sigma_m)x, \qquad x \in \mathfrak{X}.$$

Proof. We wish to show that $\lim_{m\to\infty} E(\sigma_m-\sigma)x=0$ for all x. Thus we may pass without loss of generality from consideration of the sequence $\{\sigma_m\}$ to consideration of the sequence $\{\sigma_m - \sigma\}$; that is, we may and shall assume without loss of generality that σ is void. Since $E(\sigma_m) = E(\sigma_m \sigma(T))$, by the preceding lemma, we may also assume that $\sigma_m \subseteq \sigma(T)$.

Suppose that our assertion is false, so that there is a $p > 0$ and a vector x such that $|E(\sigma_m)x| \geqq p$ for arbitrarily large m. Passing without loss of generality to a subsequence, we may assume that $|E(\sigma_m)x| \geqq p$ for all m.

For σ in $S_2(T)$ let $M(\sigma) = \sup_{\mu \subseteq \sigma} |E(\mu)x|$, $\mu \in S_2(T)$. It is clear that if $\nu_1 \subseteq \nu_2$ then $M(\nu_1) \leqq M(\nu_2)$. Let $\mu \subseteq \nu_1 \cup \nu_2$. Since $|E(\mu)x| = |E(\mu\nu_1)x + E(\mu\nu_1' \nu_2)x| \leqq M(\nu_1) + M(\nu_2)$, it follows immediately that

$$M(\nu_1 \cup \nu_2) \leqq M(\nu_1) + M(\nu_2).$$

Since

$$E(\mu)x = |E(\mu)E(\sigma)x| \leqq K|E(\sigma)x|,$$

for $\mu \subseteq \sigma$ it is seen that

$$M(\sigma) \leqq K|E(\sigma)x|.$$

Since σ_m is in $S(T)$, we can find closed sets μ_m and ν_m in $S_2(T)$ such that $\mu_m \subseteq \sigma_m$, $\nu_m \subseteq \sigma'_m$, and

$$|E(\mu_m \cup \nu_m)'x| \leqq pK^{-1}2^{-m-1}.$$

Then

$$M((\mu_m \cup \nu_m)') \leqq p2^{-m-1},$$

so that, putting $\delta_m = \sigma_m - \mu_m$, we have $\delta_m = \sigma_m \mu'_m \subseteq \mu'_m \cap \nu'_m = (\mu_m \cup \nu_m)'$ and so

$$M(\delta_m) \leqq p2^{-m-1}.$$

It follows that no finite sum $\delta_1 \cup \cdots \cup \delta_n$ can cover σ_n. Indeed,

$$M(\delta_1 \cup \cdots \cup \delta_n) \leqq \sum_{i=1}^{n} p2^{-i-1} \leqq 1/2^p,$$

while $|E(\sigma_n)x| \geqq p$. Hence

$$\sigma_n \mu_1 \mu_2 \cdots \mu_n = \sigma_n - \bigcup_{i=1}^{n} \sigma_n \delta_i$$

is nonvoid. Since $\bigcap_{i=1}^{n} \mu_i$ is a decreasing sequence of nonvoid closed subsets of the compact set $\sigma(T)$, we have $\bigcap_{i=1}^{n} \mu_i \neq \phi$. Thus, since

$\mu_i \subseteq \sigma_i$ it follows that $\bigcap_{i=1}^{n} \sigma_i \neq \phi$, contrary to assumption. Q.E.D.

The following three theorems summarize the results of this section and at the same time lay the foundation for the studies to be made in §4 that follows.

THEOREM 11. (A, B) *Let T be a bounded linear operator in the complex B-space $\mathfrak{X}$. Then there is a unique spectral measure on the field $S(T)$ with the properties*

$$\begin{aligned} E(\delta)x &= x, \qquad \delta \in S(T), \qquad \sigma(x) \subseteq \delta, \\ &= 0. \qquad \delta \in S(T), \qquad \sigma(x) \subseteq \delta'. \end{aligned}$$

This spectral measure is bounded, countably additive on $S(T)$, and commutes with T.

PROOF. From Definitions 1, 4 and 7 it is seen that $S(T) \subseteq S_1(T)$, and thus for each δ in $S(T)$ there is, by Lemma 2, one and only one projection $E(\delta)$ with $E(\delta)x = x$ if $\sigma(x) \subseteq \delta$ and $E(\delta)x = 0$ if $\sigma(x) \subseteq \delta'$. Lemma 2 also shows that $|E(\delta)|$ is bounded in δ. Lemma 3 shows that $E(\delta)$ commutes with T and Lemmas 6, 8 and 10 show that E is a countably additive spectral measure on $S(T)$. Q.E.D.

Since the field $S(T)$ is not necessarily a σ-field it is natural to ask whether or not the spectral measure E may be extended to the σ-field generated by $S(T)$. The following definition and theorems are concerned with this question.

DEFINITION 12. The symbol $M(T)$ will be used for the σ-complete Boolean algebra (or σ-field) determined by the Boolean algebra $S(T)$. The sets in $M(T)$ are called *sets measurable T* or *T-measurable sets.*

THEOREM 13. (A, B) *Let T be a bounded linear operator in the complex B-space $\mathfrak{X}$ and let E be the associated spectral measure whose existence was established in Theorem 11. Then, in the conjugate space $\mathfrak{X}^*$, there is a unique extension of the adjoint E^* to a spectral measure on the σ-field $M(T)$ of sets measurable T which is countably additive on $M(T)$ in the $\mathfrak{X}$ topology of $\mathfrak{X}^*$. This unique extension is bounded and commutes with T^*.*

PROOF. For every x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$ there is, according to the Hahn theorem, a unique countably additive extension $m(e, x, x^*)$ of the $x^*E(e)x$ from $S(T)$ to $M(T)$. From its uniqueness it is seen that $m(e, x, x^*)$ is bilinear in x, x^* and from the boundedness of $|E(e)|$ follows the boundedness of $m(e, x, x^*)$. Thus for each e in $M(T)$ there is a uniquely defined bounded linear operator $A(e)$ in $\mathfrak{X}^*$ for which

$$xA(e)x^* = m(e, x, x^*).$$

It will next be shown that the mapping $e \rightarrow A(e)$ of $M(T) \rightarrow B(\mathfrak{X}^*)$ is a spectral measure. It clearly preserves finite disjoint unions, takes complements into complements, is countably additive in the $\mathfrak{X}$ topology of $\mathfrak{X}^*$, and is bounded. It remains only to show that

$$A(\sigma)A(\delta) = A(\sigma\delta).$$

It is seen, by using the above remarks, that for a fixed σ the family of δ for which the equation is valid is a σ-field. Thus if σ is in $S(T)$ the equation holds for all δ in $M(T)$. Analogously, if δ is fixed in $M(T)$, then since the equation holds for σ in a σ-field containing $S(T)$, it must hold for all σ in $M(T)$.

Since T and $E(\delta)$ commute and since $A(\delta) = E(\delta)^*$ for δ in $S(T)$ we have

$$xT^*A(\delta)x^* = xA(\delta)T^*x^*, \qquad x \in \mathfrak{X},\ x^* \in \mathfrak{X}^*,\ \delta \in S(T).$$

Since A is countably additive in the $\mathfrak{X}$ topology of $\mathfrak{X}^*$ this identity holds for every δ in the σ-field determined by $S(T)$ and this proves that $A(\delta)$ commutes with T^* for every T-measurable set δ. Since $m(e, x, x^*) = x^*A(e)x$ is bounded in e it follows from the principle of uniform boundedness that $|A(e)|$ is bounded for e in $M(T)$. Q.E.D.

THEOREM 14. (A, B) *Let T be a bounded linear operator in the weakly complete complex B-space $\mathfrak{X}$ and let E be the associated spectral measure whose existence was established in Theorem* 11. *Then there is a uniquely determined extension of E to a spectral measure on the σ-field $M(T)$ of sets measurable T which is countably additive on $M(T)$ in the strong operator topology. This extension is bounded and commutes with T.*

PROOF. Let A be the spectral measure in the adjoint space $\mathfrak{X}^*$ which is associated with T^* as in the preceding theorem. If $\mathfrak{X}$ is weakly complete, $A(\delta)$ is the adjoint of an operator $E(\delta)$ in $\mathfrak{X}$. To see this, note that the family of all sets δ for which there exists an operator $E(\delta)$ in $\mathfrak{X}$ with $A(\delta) = E(\delta)^*$ contains $S(T)$. This family is also a Boolean algebra since A is a spectral measure. Since $\mathfrak{X}$ is weakly complete it is a σ-complete Boolean algebra and hence coincides with $M(T)$. Since $E^*(\delta)$ commutes with T^* it follows that $E(\delta)$ commutes with T for every δ in $M(T)$. Theorem 13, together with the Orlicz-Pettis theorem, shows that E is countably additive on $M(T)$ in the strong operator topology. The boundedness of E follows from that of A. Q.E.D.

3(C). **Consequences of the conditions** (A, B, C): **necessary and sufficient conditions for spectral operators.** Theorem 14 of Part 3(B) falls short of proving that T is a spectral operator in two ways. First of all the spectral measure E is not necessarily a resolution of the identity for T, for, even though it commutes with T it may not satisfy the inclusion relation $\sigma(T|E(\delta)\mathfrak{X})\subseteq\bar{\delta}$. Secondly, the field $S(T)$ or the σ-field $M(T)$ may not contain all Borel sets. The first of these difficulties will be eliminated by the hypothesis (C) to be made presently. The second of these difficulties leads us to consider operators which are spectral relative to a field other than the field of Borel sets. Such operators are described as follows:

DEFINITION 1. Let Σ be a field of sets in the complex plane and let T be a linear operator in the complex B-space $\mathfrak{X}$. Then a spectral measure E on Σ is said to be a *resolution of the identity for* T if it commutes with T and satisfies

$$\sigma(T\,|\,E(\delta)\mathfrak{X}) \subseteq \bar{\delta}, \qquad \delta \in \Sigma,$$

where $T|E(\delta)\mathfrak{X}$ is the restriction of T to $E(\delta)\mathfrak{X}$. The operator T is said to be a spectral operator of class $(\Sigma, \mathfrak{X}^*)$ if it has a bounded resolution of the identity on Σ for which the set functions $x^*E(\cdot)x$ with x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$ are all countably additive on Σ. An operator in $\mathfrak{X}^*$ is said to be a *spectral operator of class* $(\Sigma, \mathfrak{X})$ if it has a bounded resolution of the identity A on Σ for which the set functions $xA(\cdot)x^*$ with x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$ are all countably additive on Σ.

Thus T is a spectral operator if and only if it is a spectral operator of class $(B, \mathfrak{X}^*)$, where B is the field of Borel sets in the plane.

Besides the conditions (A) and (B) the following condition (C) will be assumed in most of what follows. However, when any of the assumptions (A), (B), (C) are made in a lemma or theorem they will be indicated parenthetically.

(C) *For every closed set* δ *of complex numbers the set of all vectors* x *with* $\sigma(x)\subseteq\delta$ *is also closed.*

LEMMA 2. (A, B, C) *For every set* δ *in* $S_1(T)$ *and every vector* z *in* $\mathfrak{X}$ *we have* $\sigma(E(\delta)z)\subseteq\bar{\delta}\sigma(z)$.

PROOF. Since δ is in $S_1(T)$, an arbitrary vector z in $\mathfrak{X}$ is the limit of a sequence $z_n=x_n+y_n$ with $\sigma(x_n)\subseteq\delta$ and $\sigma(y_n)\subseteq\delta'$. Thus

$$\sigma(E(\delta)z_n) = \sigma(x_n) \subseteq \bar{\delta}$$

and since $E(\delta)z_n\rightarrow E(\delta)z$, it follows from (C) that

$$\sigma(E)(\delta)z) \subseteq \bar{\delta}.$$

Since $E(\delta)$ commutes with T (cf. Lemma 3 of Part 3(B)), it is seen from Lemma 4 of Part (A) that $\sigma(E(\delta)z)\subseteq\sigma(z)$. Thus $\sigma(E(\delta)z)\subseteq\bar{\delta}\sigma(z)$. Q.E.D.

LEMMA 3. (A, B, C) *For* δ *in* $S_1(T)$ *let* $T|E(\delta)\mathfrak{X}$ *be the restriction of* T *to* $E(\delta)\mathfrak{X}$. *Then*

$$\sigma(T \mid E(\delta)\mathfrak{X}) \subseteq \bar{\delta}, \qquad \delta \in S_1(T).$$

PROOF. It follows from Lemma 3 of Part 3(B) that T commutes with $E(\delta)$ and so T maps $E(\delta)\mathfrak{X}$ into itself. It is therefore meaningful to speak of the spectrum of the restriction of T to $E(\delta)\mathfrak{X}$.

Let $\xi \notin \bar{\delta}$. It will first be shown that $\xi I - T$ is one-to-one on $E(\delta)\mathfrak{X}$. If x is in $E(\delta)\mathfrak{X}$ and $(\xi I - T)x = 0$, then, since

$$R(\lambda; T) = \sum_{n=0}^{\infty} \frac{(T - \xi I)^n}{(\lambda - \xi)^{n+1}},$$

for all large λ, it is seen that $x(\lambda) = x/(\lambda - \xi)$ for $\lambda \neq \xi$. Thus the spectrum $\sigma(x)$ contains at most the point ξ and therefore $\bar{\delta}\sigma(x)$ is void. Since $x = E(\delta)x$ it follows from Lemma 2 that $\sigma(x)$ is void and from Lemma 2 of Part 3(A) that $x = 0$. This shows that $\xi I - T$ is one-to-one on $E(\delta)\mathfrak{X}$.

It will be shown that $(\xi I - T)E(\delta)\mathfrak{X} = E(\delta)\mathfrak{X}$. Let $x = E(\delta)x$ be an arbitrary point in $E(\delta)\mathfrak{X}$. Then, by Lemma 2, $\sigma(x) \subseteq \bar{\delta}$ and so $\xi \in \rho(x)$. Thus $(\xi I - T)x(\xi) = x$ and hence $(\xi I - T)E(\delta)x(\xi) = E(\delta)x = x$ which shows that $(\xi I - T)E(\delta)\mathfrak{X} = E(\delta)\mathfrak{X}$. The operator $\xi I - T$ therefore maps $E(\delta)\mathfrak{X}$, in a one-to-one manner, onto all of itself. This means that ξ is in $\rho(T|E(\delta)\mathfrak{X})$ and thus $\sigma(T|E(\delta)\mathfrak{X}) \subseteq \bar{\delta}$. Q.E.D.

THEOREM 4. *A spectral operator* T *has the properties* (A), (B), *and* (C). *Conversely, if the bounded linear operator* T *has these properties it is a spectral operator of class* $(S(T), \mathfrak{X}^*)$. *Moreover,* T *has a resolution of the identity which is countably additive in the strong operator topology.*

PROOF. We shall only demonstrate here the sufficiency of the conditions. Let the bounded linear operator T satisfy the conditions (A), (B), and (C). Then by Theorem 11 of Part 3(B) and Lemma 3, T is a spectral operator of class $(S(T), \mathfrak{X}^*)$ with a resolution of the identity which is countably additive in the strong operator topology. Q.E.D.

THEOREM 5. *Let* T *be a bounded linear operator in a weakly complete space. Then* T *is a spectral operator if and only if* T *satisfies conditions* (A), (B), (C), *and the following condition* (D):

(D) *Every complex number is interior to a set of arbitrarily small diameter belonging to* $S(T)$.

PROOF. It follows from the preceding theorem that a spectral operator has properties (A) through (C). To show that a spectral operator T has property (D), let δ be a closed set in the complex plane, and let $\{\delta_n\}$ be an increasing sequence of closed sets whose union is the complement δ' of δ. Let E be the spectral resolution of T. Then

$$x = \lim_n \{E(\delta)x + E(\delta_n)x\}.$$

But $\sigma(E(\delta)x) \subseteq \delta$ and $\sigma(E(\delta_n)x) \subseteq \delta_n$. This shows that δ is in $S_1(T)$. Lemma 2 shows that δ is in $S_2(T)$ and, since δ_n is closed, the above equation proves that δ is in $S(T)$. Thus $S(T)$ contains every closed set and property (D) is evident.

Conversely, if the operator T satisfies conditions (A) through (D), then, by Theorem 4, it is a spectral operator of class $(S(T), \mathfrak{X}^*)$. According to Theorem 14 of Part 3(B) the resolution of the identity for T has a unique extension to a countably additive spectral measure E on $M(T)$. It will next be shown that $M(T)$ contains all Borel sets.

To do this let U be an open set of the complex plane, and let K be a compact subset of U. Then, by (D), each point p in K is interior to a certain set σ_p in $S(T)$ with $\sigma_p \subseteq U$. Since K is compact, it is contained in the union σ of a finite collection of the sets σ_p. Thus, we have shown that if K is a compact subset of U, there exists a set $\sigma \in S(T)$ such that $K \subseteq \sigma \subseteq U$. Since U is the union of a countable infinity of its own compact subsets, it follows U is in $M(T)$. Since $M(T)$ contains all open sets, it contains the family B of all Borel sets.

To complete the proof it will suffice to show that $\sigma(T|E(\delta)\mathfrak{X}) \subseteq \bar{\delta}$ for every Borel set δ. If $\lambda \notin \bar{\delta}$, then, using (D), the compact set $\bar{\delta}\sigma(T)$ may be covered by a set σ in the field $S(T)$ with $\lambda \notin \bar{\sigma}$. Since T is a spectral operator of class $(S(T), \mathfrak{X}^*)$, we have $\sigma(T|E(\sigma)\mathfrak{X}) \subseteq \bar{\sigma}$ and consequently λ is in $\rho(T|E(\sigma)\mathfrak{X})$ which means that $\lambda I - T$ is a one-to-one map of $E(\sigma)\mathfrak{X}$ into all of itself. Since $\sigma \supseteq \delta\sigma(T)$, we have $E(\sigma) \supseteq E(\delta\sigma(T)) = E(\delta)$ and consequently $E(\delta)\mathfrak{X}$ is an invariant subspace of $E(\sigma)\mathfrak{X}$. Thus $\lambda I - T$ is a one-to-one map of $E(\delta)\mathfrak{X}$ into all of itself. This proves that λ is in $\rho(T|E(\delta)\mathfrak{X})$ and thus that $\sigma(T|E(\delta)\mathfrak{X}) \subseteq \bar{\delta}$. Q.E.D.

We conclude this section with two results on adjoint operators.

LEMMA 6. *Let Σ be a field of sets in the complex plane and let T be a spectral operator of class* $(\Sigma, \mathfrak{X}^*)$. *Then its adjoint T^* is a spectral operator of class* $(\Sigma, \mathfrak{X})$.

PROOF. Let E be a resolution of the identity for T. Then the mapping $\sigma \rightarrow E^*(\sigma)$ of Σ into $B(\mathfrak{X}^*)$ is a spectral measure in $\mathfrak{X}^*$. Moreover, $xE^*(\sigma)x^*$ is evidently countably additive on Σ for each $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$. Let $\lambda \notin \bar{\sigma}$. Then the restriction of $\lambda I - T$ to $E(\sigma)\mathfrak{X}$ has an inverse R_σ. Define the operator P_σ in $\mathfrak{X}$ by putting $P_\sigma = R_\sigma E(\sigma)$. Then clearly $E(\sigma)P_\sigma = P_\sigma = P_\sigma E(\sigma)$. Hence

$$P_\sigma^* E(\sigma)^* = E(\sigma)^* P_\sigma^*,$$

so that P_σ^* maps $E(\sigma)^*\mathfrak{X}^*$ into itself. Also $(\lambda I - T)P_\sigma = E(\sigma)$, and $P_\sigma(\lambda I - T) = P_\sigma E_\sigma(\lambda I - T) = P_\sigma(\lambda I - T)E(\sigma) = E(\sigma)$. Thus

$$P_\sigma^*(\lambda I^* - T^*) = (\lambda I^* - T^*)P_\sigma^* = E(\sigma)^*.$$

Consequently the restriction of P_σ^* to $E(\sigma)^*\mathfrak{X}^*$ is the inverse of the restriction of $\lambda I^* - T^*$ to $E(\sigma)^*\mathfrak{X}^*$. Hence λ is in the resolvent of the restriction $(T^*)_\sigma$ of T^* to $E^*(\sigma)\mathfrak{X}^*$. This shows that $\sigma((T^*)_\sigma) \subseteq \bar{\sigma}$ and completes the proof. Q.E.D.

THEOREM 7. (A, B, C, D) *Let T be a bounded linear operator in the complex B-space $\mathfrak{X}$ and let B be the field of Borel sets in the plane. Then T^* is a spectral operator of class $(B, \mathfrak{X})$.*

PROOF. In view of condition (D) we have $B \subseteq M(T)$. By Theorem 13 of Part 3(B) the spectral measure E^* of the preceding lemma may be extended from $S(T)$ to a spectral measure defined on $M(T)$. Then, as in the proof of Theorem 5, it may be shown that $\sigma(T|E(\delta)\mathfrak{X}) \subseteq \bar{\delta}$ for each δ in $M(T)$. Q.E.D.

4. Operators whose spectra lie in a Jordan curve.

In the preceding section it was seen that operators satisfying the conditions (A), $\cdots$, (D) are spectral operators. In this section it will be shown that, in certain important special cases, all of these conditions, except possibly the boundedness condition (B), are automatically satisfied. Thus for the special types of operators, condition (B) becomes the condition which is necessary as well as sufficient for the operator to be a spectral operator.

LEMMA 1. *The condition* (A) *is satisfied if the spectrum of T is nowhere dense in the complex plane.*

PROOF. If the resolvent set is dense then any two analytic, or even continuous, extensions of $R(\lambda; T)x$ must coincide on their common domain of continuity. Q.E.D.

All of the special type operators to be considered in the present section will have nowhere dense spectra so that, according to Lemma 1, the condition (A) will be satisfied by all of the operators that will be studied here.

The following theorem, which applies in particular to compact operators, gives a topological restriction on the spectrum of T which guarantees that (A), (C), and (D) are all satisfied.

THEOREM 2. *If the spectrum of an operator in a weakly complete space is totally disconnected then it is a spectral operator if and only if the boundedness condition* (B) *is satisfied.*

PROOF. Let T be a bounded linear operator in the weakly complete B-space $\mathfrak{X}$. To prove the theorem it will, in view of Theorem 5 of Part 3(C), suffice to show that T has the properties (A), (C), and (D). Since the spectrum $\sigma(T)$ of T is totally disconnected it is nowhere dense and, according to Lemma 1, condition (A) is satisfied.

We recall[10] that to each *spectral set* (a spectral set is one which is an open and closed subset of the spectrum $\sigma(T)$ in its relative topology) is associated a projection $E(\delta)$ with $\sigma(T|E(\delta)\mathfrak{X})=\delta$ from which it follows that every spectral set is in $S_2(T)$. Since spectral sets are closed they are also contained in $S(T)$. Since the spectrum is totally disconnected, every spectral point is contained in a spectral set of arbitrarily small diameter and thus in an $S(T)$ set of arbitrarily small diameter. Since it is clear that every subset of the resolvent set is an $S(T)$ set, condition (D) is immediate.

To verify condition (C), let δ be a closed set of complex numbers and let

$$M(\delta) = \{x \mid \sigma(x) \subseteq \delta\}.$$

Condition (C) will be proved by showing that $M(\delta)$ is closed. Since $\sigma(x)\subseteq\sigma(T)$ we have $M(\delta)=M(\delta\sigma(T))$, and it may therefore be assumed, without loss of generality, that $\delta\subseteq\sigma(T)$. Since $\sigma(T)$ is totally disconnected, the closed set δ is an intersection $\bigcap_\alpha \delta_\alpha$ of spectral sets δ_α. Now clearly

$$M(\delta) = M\left(\bigcap_\alpha \delta_\alpha\right) = \bigcap_\alpha M(\delta_\alpha),$$

and so to see that $M(\delta)$ is closed it will suffice to see that $M(\delta_\alpha)$ is closed. Since δ_α is a spectral set, it follows that $M(\delta_\alpha)=E(\delta_\alpha)\mathfrak{X}$ and hence is closed. Q.E.D.

Theorem 2 suggests that the difficulties which may be encountered in verifying the conditions (C) and (D) are, in some way, related to the presence of connected components of the spectrum. The remainder of the present section will be devoted to a study of the case where the spectrum is contained in a finite disjoint union of connected sets each one of which is a Jordan arc. Before passing to the details of this study the following result will be introduced to allow us, without any loss of generality, to study the case where the whole spectrum is contained in one Jordan arc.

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[10] Here we are using the operational calculus on analytic functions as developed in [7] and [8].

THEOREM 3. *Let T be an operator in the B-space $\mathfrak{X}$. If $\mathfrak{X}$ is the direct sum of two of its closed subspaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, each invariant under T, and if the restrictions of T to $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are both spectral operators then T is a spectral operator.*

PROOF. Let E_1 and E_2 be the spectral resolutions of the restrictions of T to $\mathfrak{X}_1$ and $\mathfrak{X}_2$ respectively. If $x=x_1+x_2$ with $x_i \in \mathfrak{X}_i$ for $i=1, 2$, we define, for every Borel set e of complex numbers, the operator

$$E(e)x = E_1(e)x_1 + E_2(e)x_2.$$

It is clear that E is a countably additive spectral measure and that all the projections $E(e)$ commute with T. If λ is a complex number not in the closure of the set e, then $\lambda I - T$ maps each of the spaces $E_1(e)\mathfrak{X}_1$ and $E_2(e)\mathfrak{X}_2$ in a one-to-one manner onto all of itself. Hence $\lambda I - T$ maps $E(e)\mathfrak{X}$ to a one-to-one manner onto all of itself. This shows that $\sigma(T | E(e)\mathfrak{X}) \subseteq \bar{e}$ and proves that E is a spectral resolution for T and that T is a spectral operator. Q.E.D.

In most of the remainder of the present section it will be assumed that the spectrum $\sigma(T)$ of T is contained in a closed Jordan curve Γ_0. In order to avoid technical complications it will be convenient to assume also that Γ_0 is smoothly imbedded in a one-parameter family of closed rectifiable Jordan curves. More specifically, and as a basis for the analytical discussion that follows, it will be assumed that there is a function $\xi = \xi(t, \delta)$ which is twice continuously differentiable on its domain $-1 \leqq t, \delta < 1$ of definition and which has the following properties. The equation $\xi(-1, \delta) = \xi(+1, \delta)$ holds for all δ in the interval $-1 \leqq \delta \leqq 1$, whereas $\xi(s, \delta) \neq \xi(t, \delta)$ unless $s = t$ or the pair s, t is the pair $-1, 1$. Thus $\xi(\cdot, \delta)$ is the parametric representation of a simple closed rectifiable Jordan curve Γ_δ. It is assumed that the curves Γ_δ are mutually disjoint, that Γ_{δ_1} lies inside Γ_{δ_2} if $-1 \leqq \delta_1 < \delta_2 \leqq 1$, and that Γ_0 contains the spectrum $\sigma(T)$. There will be occasion to integrate around the curve Γ_δ with respect to its arc length, and for this reason it is supposed that the curves Γ_δ are oriented in the positive sense customary in the theory of complex variables. The simple Jordan arc Δ_{λ_0} which is parametrized by the function $\xi(t_0, \cdot)$ is called the *transversal through the point* $\lambda_0 = \xi(t_0, 0)$. The principal assumption that will be made throughout most of this section is that the resolvent $R(\lambda; T)$ has a finite rate of growth as λ approaches a spectral point λ_0 along the transversal Δ_{λ_0} through λ_0. This rate of growth hypothesis is stated formally as follows.

(G) The spectrum of T is contained in the rectifiable Jordan curve Γ_0 described above. Moreover, for each spectral point λ_0 there are two positive integers $\nu=\nu(\lambda_0)$ and $M=M(\lambda_0)$ depending upon λ_0 and such that

$$\left| (\lambda - \lambda_0)^\nu R(\lambda; T) \right| \leqq M, \qquad \lambda \neq \lambda_0, \lambda \in \Delta\lambda_0.$$

Although the rate of growth condition (G) will be assumed in most of what follows, it will be stated either explicitly or parenthetically (as was done with the conditions (A), · · · , (D)) in any theorem where it is used. It will be seen that this growth condition implies the conditions (A) and (C) and that the boundedness and growth conditions (B) and (G) together come very near to insuring that the operator T is a spectral operator.

Lemma 4. *An operator with property* (G) *also has properties* (A) *and* (C).

Proof. If the operator T in the B-space $\mathfrak{X}$ satisfies the growth condition (G), its spectrum lies in the rectifiable Jordan curve Γ_0. Thus the spectrum is nowhere dense and condition (A) follows from Lemma 1.

To prove (C), let δ be a closed subset of the complex plane and let

$$\mathfrak{M}(\delta) = \left\{ x \,\middle|\, x \in \mathfrak{X}, \sigma(x) \subseteq \delta \right\}.$$

It will be shown that $M(\delta)$ is closed. For every x we have $\sigma(x) \subseteq \sigma(T) \subseteq \Gamma_0$ and thus $M(\delta)=M(\delta\Gamma_0)$ which allows us to assume, with no loss of generality, that δ is a subset of the curve Γ_0. The set δ is therefore an intersection $\delta=\bigcap_\alpha \delta_\alpha$ of sets δ_α each one of which is the complement in Γ_0 of an open subinterval of Γ_0. Since

$$M(\delta) = M\left(\bigcap_\alpha \delta_\alpha \right) = \bigcap_\alpha M(\delta_\alpha),$$

in order to see that $M(\delta)$ is closed it will suffice to prove that $M(\delta_\alpha)$ is closed. In other words, we may and shall assume that δ is the complement of an open subinterval γ of Γ_0. Let $\{x_n\}$ be a sequence in $\mathfrak{X}$, convergent to the point x, and with $\rho(x_n) \supseteq \gamma$. To prove (C) it will be shown that $\rho(x) \supseteq \gamma$. To do this it is evidently sufficient to show that $\rho(x)$ contains an arbitrary open subinterval γ_0 of γ. Let a and b be the end points of γ_0 and let C be a simple Jordan curve composed of the transversals Δ_a, Δ_b and arcs connecting their end points in such a way that C includes γ_0 in its interior, intersects Γ_0 only at the points

a, b, and includes the rest of Γ_0 in its exterior.

The condition (G) shows that there is an integer N such that

$$\lim_{\lambda\to a;\lambda\in C} (\lambda - a)^N(\lambda - b)^N x_n(\lambda) = \lim_{\lambda\to b;\lambda\in C} (\lambda - a)^N(\lambda - b)^N x_n(\lambda) = 0$$

uniformly in $n=1, 2, \cdots$. Thus there are open subarcs N_a, N_b of C containing a, b respectively and such that the vector $y_n(\lambda) = (\lambda-a)^N(\lambda-b)^N x_n(\lambda)$ has norm

$$|y_n(\lambda)| < \epsilon/2, \qquad n \geqq 1, \qquad \lambda \in N_a \cup N_b.$$

Since $x_n \to x$ we have

$$\lim_{n\to\infty} y_n(\lambda) = (\lambda - a)^N(\lambda - b)^N R(\lambda; T)x$$

uniformly for λ in $C-N_a-N_b$. This fact together with the preceding inequality shows that for some integer n_0 depending upon ϵ we have

$$|y_n(\lambda) - y_m(\lambda)| < \epsilon, \qquad \lambda \in C,\ n, m \geqq n_0,$$

and thus proves that the sequence $\{y_n(\lambda)\}$ converges uniformly for λ in C. By the maximum modulus principle this sequence converges uniformly on the union of C and its interior to an analytic function $y(\lambda)$. Since

$$y(\lambda) = \lim_n y_n(\lambda) = (\lambda - a)^N(\lambda - b)^N x(\lambda)$$

for $\lambda \notin \Gamma_0$ the vector

$$X(\lambda) = \frac{y(\lambda)}{(\lambda - a)^N(\lambda - b)^N}$$

is an analytic continuation of $x(\lambda)$ into the interior of C. Since $(\lambda I-T)x(\lambda)=x$ for $\lambda\notin\Gamma_0$, it follows that $(\lambda I-T)X(\lambda)=x$ for all λ interior to C. Thus $\rho(x)$ includes the interior of C and, since the interior of C includes γ_0, the proof is complete. Q.E.D.

According to the preceding lemma and Theorem 5 of §3(C), an operator T in a weakly complete space will be spectral if it satisfies (B), (G) and (D). We shall now study the condition (D) more carefully and see that the conditions (B) and (G) come very near to implying the condition (D). This will allow us to replace the condition (D), and in a variety of ways, by more satisfactory conditions. For example, it will be seen that an operator T in a reflexive space which satisfies (G) and whose adjoint satisfies (B) is a spectral operator.

Before starting this study it will be convenient to restate the condition (G) in a form more suitable to the analysis that follows. In the first place, it is clear that the integers $\nu=\nu(\lambda_0)$ and $M=M(\lambda_0)$ with the required properties exist for every λ_0 in Γ_0 even if λ_0 is not in the spectrum. Also, it may be assumed that for each λ_0 in Γ_0 the transversal Δ_{λ_0} lies within the circle of radius 1/2 and center λ_0. This shortening of the transversal Δ_{λ_0} may be achieved by replacing $\xi(\lambda, \delta)$ by $\xi_1(\lambda, \delta_1)$ where $\delta_1=K\delta$ with K sufficiently large. Now, if every point of Δ_{λ_0} is within a distance of 1/2 from λ_0, it follows from (G) that

$$\lim_{N\to\infty} (\lambda - \lambda_0)^N R(\lambda; T) = 0$$

uniformly for λ in Δ_{λ_0}. Thus there is an integer valued function $\nu=\nu(\lambda_0)$ defined for every λ_0 in Γ_0 and such that $|(\lambda-\lambda_0)^{\nu(\lambda_0)}R(\lambda; T)| \leqq 1$ for every λ in Δ_{λ_0} except $\lambda=\lambda_0$. In other words the function $M=M(\lambda_0)$ of condition (G) may, without loss of generality, be assumed to be identically one. Thus, condition (G) may be restated in the following equivalent form.

(G) The spectrum of T is contained in the rectifiable Jordan curve Γ_0. Moreover, there is an integer valued function ν defined on Γ_0 such that for every λ_0 in Γ_0,

$$|(\lambda - \lambda_0)^{\nu(\lambda_0)} R(\lambda; T)| \leqq 1, \qquad \lambda \neq \lambda_0,\ \lambda \in \Delta_{\lambda_0}.$$

In the analysis to follow, it is this inequality that will be used rather than the one in the earlier formulation of the growth condition (G).

Definition 5. An integer valued function ν defined on Γ_0 and satisfying the preceding inequality is called an *index function* for T. An *interval of constancy* relative to T is a nonvoid open subinterval of Γ_0 upon which some index function for T is constant. A point λ in Γ_0 is said to be *regular relative* to T if it belongs to an interval of constancy and if, in addition, there is an integer n such that the manifold

$$(\lambda I - T)^n \mathfrak{X} + \{x \mid (\lambda I - T)^n x = 0\}$$

is dense in $\mathfrak{X}$.

It should be noted that if

$$\mathfrak{X} = \mathrm{Cl}(T - \lambda I)^n \mathfrak{X} + \{x \mid (T - \lambda I)^n x = 0\},$$

then, by applying the operator $(T-\lambda I)^n$ to both sides of this equation, one obtains the inclusion relation

$$(T - \lambda I)^n \mathfrak{X} \subseteq \mathrm{Cl}(T - \lambda I)^{2n} \mathfrak{X}.$$

Thus, since the manifolds $(T-\lambda I)^m\mathfrak{X}$ decrease as m increases, it follows that

$$(T - \lambda I)^n\mathfrak{X} \subseteq \mathrm{Cl}(T - \lambda I)^{n+1}\mathfrak{X}.$$

Since the manifolds $\{x \mid (T-\lambda I)^m x=0\}$ increase with m, it follows that

$$(T - \lambda I)^{n+1}\mathfrak{X} + \{x \mid (T - \lambda I)^{n+1}x = 0\}$$

is dense in $\mathfrak{X}$. By induction it is seen that the manifold

$$(T - \lambda I)^{n+k}\mathfrak{X} + \{x \mid (T - \lambda I)^{n+k}x = 0\}$$

is dense in $\mathfrak{X}$ for all $k \geqq 0$. This fact will be stated in the following lemma for future reference.

Lemma 6. *A complex number λ is regular relative to T if and only if it is contained in an interval of constancy relative to T and, for all sufficiently large integers n, the manifold*

$$(T - \lambda I)^n\mathfrak{X} + \{x \mid (T - \lambda I)^n x = 0\}$$

is dense in $\mathfrak{X}$.

It should also be noted that every point λ of Γ_0 which is in the resolvent set of T is regular relative to T. This follows since the resolvent set is open and $R(\lambda; T)$ is continuous so that λ is interior to an interval where some index function is constant. Also for λ in the resolvent set the density requirement of Definition 5 is satisfied, since for such λ, $(T-\lambda I)\mathfrak{X}=\mathfrak{X}$.

Lemma 7. (A) *If $(\lambda_0 I-T)^n x=0$ for some integer n and some $x \neq 0$ then $\sigma(x)=\{\lambda_0\}$.*

Proof. Since it is finite, the series

$$\begin{aligned} X(\lambda) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(\lambda - \lambda_0)^{j+1}} (\lambda_0 I - T)^j x \\ &= \sum_{j=0}^{n} \frac{(-1)^j}{(\lambda - \lambda_0)^{j+1}} (\lambda_0 I - T)^j x \end{aligned}$$

converges for every $\lambda \neq \lambda_0$ and satisfies the equation $(\lambda I-T)X(\lambda)=x$. Thus $X(\lambda)$ is an analytic extension of $R(\lambda; T)x$ to the complement of $\{\lambda_0\}$. This means that $\sigma(x) \subseteq \{\lambda_0\}$. Since, by Lemma 2 of §3(A), $\sigma(x)$ is not void, we have $\sigma(x)=\{\lambda_0\}$. Q.E.D.

Lemma 8. (B, G) *Every closed subinterval of Γ_0 whose end points are egular relative to T belongs to $S_1(T)$.*

PROOF. Let γ be a closed subinterval of Γ_0 whose end points λ_1, λ_2 are regular relative to T. It is clear that, by making a suitable change in the parameter s in the function $\xi(s, \delta)$, it may be assumed that $\lambda_1=\xi(-(1/2), 0)$ and $\lambda_2=\xi(1/2, 0)$. Since λ_1 and λ_2 are interior to intervals of constancy relative to T, there is an $\epsilon>0$ such that for $|\lambda_0-\lambda_1|<\epsilon$ or $|\lambda_0-\lambda_2|<\epsilon$ the inequality

$$\text{(i)} \qquad |\lambda - \lambda_0|^N |R(\lambda; T)| \leqq 1, \qquad \lambda_0 \neq \lambda \in \Delta_{\lambda_0}$$

holds for all sufficiently large values of N. In view of Lemma 6 the integer N may be fixed so that the inequality (i) holds and also so that the manifolds

$$\mathfrak{M}_i = (\lambda_i I - T)^N \mathfrak{X} + \{x \mid (\lambda_i I - T)^N x = 0\}, \qquad i = 1, 2,$$

are both dense in $\mathfrak{X}$. Since $\mathfrak{M}_2$ is dense in $\mathfrak{X}$ the manifold

$$(\lambda_1 I - T)^N \mathfrak{M}_2 + \{x \mid (\lambda_1 I - T)^N x = 0\}$$

is dense in $\mathfrak{X}$, so that

$$(\lambda_1 I - T)^N (\lambda_2 I - T)^N \mathfrak{X} + \{x \mid (\lambda_1 I - T)^N x = 0\} + x \mid (\lambda_2 I - T)^N x = 0$$

is also dense in $\mathfrak{X}$. By Lemma 7, $\sigma(x) \subset \gamma$ if $(\lambda_i I - T)^N x = 0$ for $i=1$ or $i=2$, so that to prove the present lemma it will be sufficient to show that every element of the form $y=(\lambda_1 I-T)^N(\lambda_2 I-T)^N x$ may be approximated arbitrarily closely by a sum z_1+z_2 where $\sigma(z_1)$ is contained in γ and $\sigma(z_2)$ is contained in the complement γ'.

Actually more will be proved, for it will be shown that z_1 and z_2 may be chosen so that z_1+z_2 is arbitrarily close to y and the spectra $\sigma(z_1)$, $\sigma(z_2)$ are contained in the *interior* of γ, γ' respectively. Using the operational calculus for analytic functions[10] we have

$$\begin{aligned} y &= (\lambda_1 I - T)^N (\lambda_2 I - T)^N x \\ \text{(ii)} \qquad &= \left\{ \int_{\Gamma_1} - \int_{\Gamma_{-1}} \right\} (\lambda_1 - \lambda)^N (\lambda_2 - \lambda)^N R(\lambda; T) x d\lambda. \end{aligned}$$

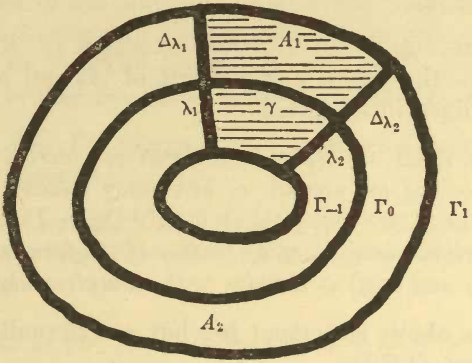
The transversals Δ_{λ_1} and Δ_{λ_2} divide the annular region between Γ_1 and Γ_{-1} into two simply connected areas, each bounded by a curve consisting of the arcs Δ_{λ_1}, Δ_{λ_2} and portions of Γ_1, Γ_{-1}. Let the area containing γ be called A_1, and its positively oriented bounding curve, C_1. Let the area containing the complement γ' be called A_2 and its positively oriented bounding curve C_2. Then, since the integrand in the expression (ii) for y is bounded on C_1 and C_2, we have

$$y = y_{C_1} + y_{C_2}$$

where

$$\text{(iii)} \qquad y_{C_i} = \frac{1}{2\pi i}\int_{C_i} (\lambda_1 - \lambda)^N(\lambda_2 - \lambda)^N R(\lambda; T)x\,d\lambda, \qquad i = 1, 2.$$

It will be shown that y_{C_1} can be approximated arbitrarily closely by vectors whose spectra lie interior to C_1, and that y_{C_2} can be approximated arbitrarily closely by elements whose spectra lie interior to C_2. The details will be given only for y_{C_2}, but the proof for y_{C_1} is quite similar.



Let $\Delta_\epsilon^-, \Delta_\epsilon^+$ be the transversals through the points $\xi((-1/2)-\epsilon, 0)$, $\xi((1/2)+\epsilon, 0)$ respectively. Let A_1^ϵ, A_2^ϵ be the two simply connected regions into which the annular region between Γ_1 and Γ_{-1} is divided by the transversals Δ_ϵ^- and Δ_ϵ^+. Let A_2^ϵ be that one of these areas contained in A_2, and let C_2^ϵ be its bounding curve. In view of (i) the integrand in (iii) is uniformly bounded, and it follows from an elementary calculation using Lebesgue's dominated convergence theorem that

$$y_{C_2} = \lim_{\epsilon \to 0} y_{C_2^\epsilon}$$

where

$$y_{C_2} = \frac{1}{2\pi i}\int_{C_2} (\lambda_1 - \lambda)^N(\lambda_2 - \lambda)^N R(\lambda; T)x\,d\lambda.$$

Thus, to see that y_{C_2} is the limit of a sequence of vectors whose spectra are interior to C_2, it will suffice to show the spectrum $\sigma(y_{C_2^\epsilon})$ is contained in A_2^ϵ. It is evident that the integral

$$I(\xi) = \frac{1}{2\pi i}\int_{C^\epsilon} (\lambda_1 - \lambda)^N(\lambda_2 - \lambda)^N(\xi - \lambda)^{-1}R(\lambda; T)x\,d\lambda$$

is analytic for ξ outside A_2^ϵ. Since

$$\begin{aligned}(\xi I - T)I(\xi) &= \frac{1}{2\pi i}\int_{C_2^\epsilon} (\lambda_1 - \lambda)^N(\lambda_2 - \lambda)^N(\xi - \lambda)^{-1}(\xi I - T)R(\lambda; T)x\,d\lambda \\ &= \frac{1}{2\pi i}\int_{C_2^\epsilon} (\lambda_1 - \lambda)^N(\lambda_2 - \lambda)^N(\xi - \lambda)^{-1}(\xi - \lambda)R(\lambda; T)x\,d\lambda \\ &= y_{C_2^\epsilon},\end{aligned}$$

it follows that $I(\xi)$ is an analytic extension of $R(\xi; T)y_{C_2^\epsilon}$. Thus $\rho(y_{C_2^\epsilon})$ includes the entire complement of A_2^ϵ and so the spectrum $\sigma(y_{C_2^\epsilon})$ is contained in A_2^ϵ. Q.E.D.

COROLLARY 9. (B, G) *Let γ be a closed subinterval of Γ_0 whose end points λ_1, λ_2 belong to intervals of constancy relative to T. Then, for sufficiently large N, the manifold $(\lambda_1 I - T)^N(\lambda_2 I - T)^N\mathfrak{X}$ is contained in the closed manifold determined by vectors of the form $z_1 + z_2$ where $\sigma(z_1)$ is interior to γ and $\sigma(z_2)$ is interior to the complementary arc γ'.*

PROOF. The above statement is what was actually proved in the preceding proof. Q.E.D.

LEMMA 10. (B, G) *If the set of points regular relative to T is dense on Γ_0 then every closed subinterval of Γ_0 whose end points are regular relative to T is in $S(T)$ and every Borel subset of the plane is measurable T.*

PROOF. Let γ be a closed subinterval of Γ_0 whose end points are regular relative to T. Since the points regular relative to T are dense in Γ_0, there is an increasing sequence $\{\gamma_n\}$ of open subintervals of Γ_0 each member of which has regular end points in the arc γ' complementary to γ and whose union is the whole arc γ'. By Lemma 8 the intervals γ and $\bar{\gamma}_n$, $n = 1, 2, \cdots$, are all in $S_1(T)$. Since γ is in $S_1(T)$ there are, for each x in $\mathfrak{X}$ and $\epsilon > 0$, vectors y and z such that

$$|y + z - x| < \epsilon, \qquad \sigma(y) \subseteq \gamma, \ \sigma(z) \subseteq \gamma'.$$

Since $\sigma(z)$ is compact we have $\bar{\gamma}_n \supseteq \gamma_n \supseteq \sigma(z)$ for all sufficiently large n. Thus, by Lemma 2 of §3(B), $[E(\gamma) + E(\bar{\gamma}_n)](y + z) = y + z$ for all large n. By the boundedness assumption (B) the norms of the projections $E(\gamma) + E(\bar{\gamma}_n)$ are bounded in n and, since $\epsilon > 0$ is arbitrary, we have

$$\text{(i)} \qquad x = \lim_n\, [E(\gamma)x + E(\bar{\gamma}_n)]x, \qquad x \in X.$$

Now it follows from Lemma 4 that T has the properties (A) and (C), and thus it is seen from Lemma 2 of §3(C) that

$$\text{(ii)} \qquad \sigma(E(\gamma)x) \subseteq \gamma\sigma(x), \qquad \sigma(E(\bar{\gamma}_n)x) \subseteq \bar{\gamma}_n\sigma(x).$$

These relations (i) and (ii) show that γ is in $S_2(T)$. Since γ is an arbitrary closed subinterval of Γ_0 whose end points are regular relative to T, this proves that the closure $\bar{\gamma}_n$ is also in $S_2(T)$. Thus it follows from (i) that γ is in $S(T)$. It follows that every Borel subset of Γ_0 is measurable T. By Lemma 9 of §3(B) every subset of $\rho(T)$ is in $S(T)$ and hence is a set measurable T. Thus, since $\sigma(T)\subseteq\Gamma_0$, every Borel set in the plane is measurable T. Q.E.D.

In view of Lemma 10 it behooves us to seek conditions under which points on Γ_0 which are regular relative to T are dense on Γ_0. Some results in this direction will be found in the next four lemmas.

Lemma 11. (G) *The union of all intervals of constancy relative to T is an open set dense in Γ_0.*

Proof. It is clear that the union of intervals of constancy is open. To see that it is dense let γ be a closed subarc of Γ_0 having positive length and let

$$\begin{aligned}\gamma_n &= \{\lambda_0 \mid \lambda_0 \in \gamma,\ |\lambda - \lambda_0|^n |R(\lambda; T)| \leqq 1, \lambda_0 \neq \lambda \in \Delta_{\lambda_0}\} \\ &= \{\lambda_0 \mid \lambda_0 \in \gamma,\ |\xi(\lambda_0, \delta) - \lambda_0|^n R(\xi(\lambda_0, \delta); T)| \leqq 1, 0 < |\delta| \leqq 1\}.\end{aligned}$$

It is clear from the second expression for γ_n that it is closed, and it follows from (G) that every point in γ is in one of the sets γ_n. Thus, by the Baire category theorem one of the sets γ_n contains a nontrivial subinterval of γ. Q.E.D.

Lemma 12. (G) *If the point spectrum of the adjoint T^* contains no nontrivial subarc of Γ_0 then the set of points regular relative to T is dense in Γ_0.*

Proof. If λ is not in the point spectrum of the adjoint of T then there is no functional $x^*\neq 0$ for which $x^*(\lambda I-T)\mathfrak{X}=0$. In view of the Hahn-Banach theorem this means that $(\lambda I-T)\mathfrak{X}$ is dense in $\mathfrak{X}$. Hence if λ is also in an interval of constancy for T then λ is regular relative to T. The present lemma thus follows from Lemma 11. Q.E.D.

Lemma 13. (G) *If the space $\mathfrak{X}$ is reflexive and if the function identically one on Γ_0 is an index function for T then every point of Γ_0 is regular relative to T.*

Proof. It is clear that Γ_0 itself is an interval of constancy relative to T. Let x be an arbitrary vector in $\mathfrak{X}$, λ_0 an arbitrary point on Γ_0 and let $\{\lambda_n\}$ be a sequence of points in the transversal Δ_{λ_0} with $\lambda_n \neq \lambda_0$ and $\lambda_0 = \lim \lambda_n$. Since $\mathfrak{X}$ is reflexive, the bounded

$$\{(\lambda_n - \lambda_0)R(\lambda_n; T)x\}$$

contains a subsequence weakly convergent to an element y of $\mathfrak{X}$. By replacing the sequence $\{\lambda_n\}$ by a suitably chosen subsequence it may therefore be assumed that, in the weak operator topology, we have

$$\lim_n (\lambda_n - \lambda_0)R(\lambda_n; T)x = y.$$

Then $(\lambda_0 I - T)y$ is the weak limit of

$$(\lambda_n - \lambda_0)(\lambda_0 I - T)R(\lambda_n; T)x = (\lambda_n - \lambda_0)x - (\lambda_0 - \lambda_n)^2 R(\lambda_n; T)x$$

and this limit is clearly zero. Thus $(\lambda_0 I - T)y = 0$.

It will next be shown that the vector $x - y$ is in the closure of the manifold $(\lambda_0 I - T)\mathfrak{X}$. To see this it will, in view of the Hahn-Banach theorem, suffice to show that $x^*(x-y) = 0$ for every linear functional x^* which vanishes on $(\lambda_0 I - T)\mathfrak{X}$. If x^* is such a functional then $T^*x^* = \lambda_0 x^*$,

$$R(\lambda_n : T)^* x^* = \frac{x^*}{\lambda_n - \lambda_0},$$

and so

$$x^*y = \lim_{n \to \infty} x^*(\lambda_n - \lambda_0)R(\lambda_n; T)x = x^*x,$$

and $x^*(x-y) = 0$. It has been shown that an arbitrary vector x in $\mathfrak{X}$ is the sum of a vector y with $(\lambda_0 I - T)y = 0$ and a vector $x - y$ in the closure of $(\lambda_0 I - T)\mathfrak{X}$. Since λ_0 is interior to an interval of constancy relative to T it is therefore a regular point relative to T. Q.E.D.

LEMMA 14. (G) *If* $\mathfrak{X}$ *is reflexive and if the adjoint* T^* *satisfies the boundedness condition* (B) *then the regular points relative to* T *are dense in* Γ_0 *and, in particular, every interval of constancy relative to* T *consists entirely of regular points.*

PROOF. In view of Lemma 11 it suffices to show that a point λ_0 in an interval of constancy relative to T is regular. Since $\sigma(T^*) = \sigma(T)$ and $R(\lambda; T^*) = R(\lambda; T)^*$, it follows that T^* also satisfies the growth condition (G) and that every index function for T is also an index function for T^* and vice versa. By applying Corollary 9 to T^* and the interval consisting of the single point λ_0 it is seen that, for N sufficiently large, every element in the manifold $(\lambda_0 I^* - T^*)^N \mathfrak{X}^*$ may be approximated by elements z^* with $\lambda_0 \notin \sigma(z^*)$. To prove that λ_0 is regular relative to T it will be shown that, for such N, the manifold

$$\text{(i)} \qquad (\lambda_0 I - T)^N \mathfrak{X} + \{x \mid \lambda_0 I - T)^N x = 0\}$$

is dense in $\mathfrak{X}$. To do this it will, in view of the Hahn-Banach theorem, suffice to show that the functional $x^* = 0$ is the only functional which vanishes on the manifold (i). Let x^* be such a functional. Then

$$\text{(ii)} \qquad (\lambda_0 I^* - T^*)^N x^* = 0.$$

To see that $x^* = 0$ it will first be shown that

$$\text{(iii)} \qquad x^* \in \mathrm{Cl}(\lambda_0 I^* - T^*)^N \mathfrak{X}^*.$$

If (iii) is not true then, since $\mathfrak{X}$ is reflexive, it follows from the Hahn-Banach theorem that there is an x in $\mathfrak{X}$ with $x^* x \neq 0$ and $[(\lambda_0 I^* - T^*)^N \mathfrak{X}^*]x = 0$ which means that $(\lambda_0 I - T)^N x = 0$. Since x^* vanishes on the manifold (i) we have $x^* x = 0$ which contradicts the inequality $x^* x \neq 0$ and establishes (iii).

Now from (ii) and (iii) together it may be concluded that $x^* = 0$. To do this note first that, according to Lemma 7, $\sigma(x^*) = \{\lambda_0\}$. By Corollary 9, applied to T^* and the interval consisting of the single point λ_0, there is a sequence $\{x_n^*\}$ converging to x^* with $\lambda_0 \notin \sigma(x_n^*)$. Since T^* satisfies the condition (B) we have

$$\left| x^* \right| \leqq K \left| x^* - x_n^* \right| \to 0,$$

and so $x^* = 0$. This completes the proof that the manifold (i) is dense in $\mathfrak{X}$ and thus proves that λ_0 is regular relative to T. Q.E.D.

The preceding lemmas, when combined with the general criteria given in Theorems 5 and 7 of §3(C), allow us to summarize a set of conditions that are sufficient to guarantee that an operator is a spectral operator. This will be done in the following two results.

THEOREM 15. *If a bounded linear operator in a weakly complete space satisfies the boundedness condition* (B) *and the growth condition* (G) *then it is a spectral operator provided that any one of the following conditions holds.*

(a) *The point spectrum of the adjoint contains no nontrivial subarc of* Γ_0.

(b) *The space is reflexive and the function*

$$\nu(\lambda) = 1, \qquad \lambda \in \Gamma_0$$

is an index function for the operator.

(c) *The space is reflexive and the adjoint operator satisfies the boundedness condition* (B).

Proof. If the bounded linear operator T in a weakly complete space has the properties (B) and (G) then, by Lemma 4, it has the properties (A) and (C). Thus, in view of Theorem 5 of §3(C), to prove the present theorem it suffices to show that T has property (D). According to Lemma 10 the condition (D) will be satisfied if the points regular relative to T are dense on Γ_0. Thus Lemmas 12, 13, 14 give the desired conclusions. Q.E.D.

Theorem 16. *Let T be a bounded linear operator in the B-space $\mathfrak{X}$ which satisfies the conditions* (B) *and* (G) *and let B be the field of Borel sets in the plane. Then the adjoint T^* is a spectral operator of class $(B, \mathfrak{X})$ provided that any one of the conditions* (a), (b), (c) *of the preceding theorem hold.*

Proof. The proof is the same as that of the preceding theorem except that Theorem 7 of §3(C) is used instead of Theorem 5.

It is to be expected from analogies with finite matrices that a spectral operator T which satisfies the growth condition (G) will be a spectral operator of type $m-1$ if and only if the constant function $\nu(\lambda)\equiv m$ is an index function. For convenience of reference this finite rate of growth condition is stated formally as follows.

Definition 17. The bounded linear operator T is said to satisfy the growth condition (G_m) if its spectrum lies in the curve Γ_0 and if for some constant M,

$$|(\xi - \xi_\delta)^m R(\xi_\delta; T)| \leqq M, \qquad \xi \in \Gamma_0,\ 0 < |\delta| \leqq 1,$$

where ξ_δ is the intersection of the transversal Δ_ξ with the curve Γ_δ.

Theorem 18. *A bounded linear operator T in Hilbert space whose spectrum lies in the Jordan curve Γ_0 will be a spectral opertor of type $m-1$ if and only if both T and its adjoint satisfy the conditions* (B) *and* (G_m).

Proof. If T is a spectral operator in the Hilbert space $\mathfrak{X}$ then, since $\mathfrak{X}$ is reflexive, the adjoint T^* is, by Lemma 6 of §3(C), a spectral operator. By Theorem 4 of §3(C), both T and T^* satisfy the boundedness condition (B). Now if T is of type $m-1$ with radical part N and resolution of the identity E then

$$R(\xi; T) = \sum_{n=0}^{m-1} N^n \int_{\sigma(T)} \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}}, \qquad \xi \in \rho(T),$$

from which it is apparent that T and consequently T^* also satisfy the condition (G_m).

Conversely, let T and its adjoint satisfy the conditions (B) and (G_m). It is clear from Theorem 15(c) that T is a spectral operator and so it suffices to prove only that T is of type $m-1$. The proof of this will require the following lemma.

LEMMA 19. *Let T be a spectral operator in Hilbert space $\mathfrak{H}$ and let E be its resolution of the identity. Then there is a constant K such that for any finite collection A_j, $j=1, 2, \cdots, n$, of bounded operators in $\mathfrak{H}$ which commute with T, and any collection σ_j, $j=1, 2, \cdots, n$, of disjoint Borel sets, we have*

$$\left| \sum_{j=1}^{n} A_j E(\sigma_j) \right| \leqq K \sup_{1 \leq j \leq n} |A_j| .$$

PROOF. According to the Lorch-Mackey-Wermer result there is a linear one-to-one map B with $B\mathfrak{H}=\mathfrak{H}$, with B and B^{-1} both continuous and such that for each Borel set σ the projection

$$P(\sigma) = BE(\sigma)B^{-1}$$

is self-adjoint. If $B_j = BA_jB^{-1}$ then

$$B\left\{ \sum_{j=1}^{n} A_j E(\sigma_j) \right\} B^{-1} = \sum_{j=1}^{n} B_j P(\sigma_j).$$

Since A_j commutes with T it commutes with $E(\sigma)$ and hence B_j commutes with $P(\sigma)$. Thus

$$\begin{aligned} \left| \sum_{j=1}^{n} B_j P(\sigma_j) x \right|^2 &= \left| \sum_{j=1}^{n} P(\sigma_j) B_j x \right|^2 = \sum_{j=1}^{n} |P(\sigma_j) B_j x|^2 \\ &\leqq \sup_j |B_j|^2 \sum_{j=1}^{n} |P(\sigma_j) x|^2 \\ &\leqq \sup_j |B_j|^2 |x|^2, \end{aligned}$$

which proves the lemma. Q.E.D.

The remaining part of the proof of Theorem 18 will involve Riemann integrals of the type

$$\text{(i)} \qquad \int_{\sigma(T)} F(\xi) E(d\xi),$$

where E is the resolution of the identity for T and F is an operator valued function defined on $\sigma(T)$, continuous in the uniform operator topology, and for which

$$\text{(ii)} \qquad F(\xi)T = TF(\xi), \qquad \xi \in \sigma(T).$$

It follows that $F(\xi)$ also commutes with the projections in the range of E, i.e.,

$$\text{(iii)} \qquad F(\xi)E(\sigma) = E(\sigma)F(\xi), \qquad \xi \in \sigma(T),$$

for every Borel set σ. If $\pi=(\sigma_1, \cdots, \sigma_n)$, $\pi'=(\sigma_1', \cdots, \sigma_{n'}')$ are two partitions of $\sigma(T)$ and if $\xi_1\in\sigma_i$, $\xi_j'\in\sigma_j'$, $i=1, \cdots, n$, $j=1, \cdots, n'$, then using (iii) and Lemma 19, it is seen that, for some constant K,

$$\begin{aligned} \left| \sum_{i=1}^{n} F(\xi_i)E(\sigma_i) - \sum_{j=1}^{n'} F(\xi_j')E(\sigma_j') \right| &= \left| \sum_{i=1}^{n} \sum_{j=1}^{n'} \{F(\xi_i) - F_j(\xi_j')\} E(\sigma_i\sigma_j') \right| \\ &\leqq K \sup \left| F(\xi_i) - F(\xi_j') \right|, \end{aligned}$$

where the supremum is taken over those i and j for which $\sigma_i\sigma_j'$ is not void. If by the norm $|\pi|$ is understood the quantity

$$|\pi| = \max_{1\leq i\leq n} \operatorname{diam} \sigma_i,$$

it is seen that the limit

$$\lim_{|\pi|\to 0} \sum_{i=1}^{n} F(\xi_i)E(\sigma_i)$$

exists in the uniform topology of operators for every function F on $\sigma(T)$ which is continuous in the uniform operator topology and satisfies (ii). This limit defines the Riemann integral (i). It is clear that the integral is linear in F and satisfies the inequality

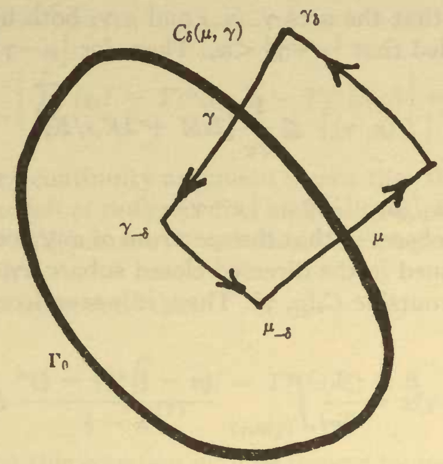
$$\text{(iv)} \qquad \left| \int_{\sigma(T)} F(\xi)E(d\xi) \right| \leqq K \sup_{\xi\in\sigma(T)} \left| F(\xi) \right| .$$

If F and G are two continuous operator valued functions both satisfying (ii) then, since

$$\left(\sum_{i=1}^{n} F(\xi_i)E(\sigma_i) \right) \left(\sum_{j=1}^{n} G(\xi_j)E(\sigma_j) \right) = \sum_{i=1}^{n} F(\xi_i)G(\xi_i)E(\sigma_i),$$

it follows that

$$\text{(v)} \qquad \left\{ \int_{\sigma(T)} F(\xi)E(d\xi) \right\} \left\{ \int_{\sigma(T)} G(\xi)E(d\xi) \right\} = \int_{\sigma(T)} F(\xi)G(\xi)E(d\xi).$$



It will next be shown that

$$\text{(vi)} \qquad \int_{\sigma(T)} (\xi I - T)^{2m} E(d\xi) = 0.$$

The proof of (vi) will use the integral

$$\text{(vii)} \qquad I(\mu, \gamma) = \frac{1}{2\pi i} \int_{C_\delta(\mu,\gamma)} (\mu - \xi)^m (\gamma - \xi)^m R(\xi; T) d\xi,$$

where μ, γ are points on Γ_0 and, for $0 < \delta \leqq 1$, the contour $C_\delta(\mu, \gamma)$ is the positively oriented contour through μ, γ which is defined as follows. For an arbitrary ξ in Γ_0 let ξ_δ be the intersection of the transversal through ξ with the curve Γ_δ. The curve $C_\delta(\mu, \gamma)$ is a positively oriented Jordan curve through the points μ_δ, γ_δ, $\gamma_{-\delta}$, $\mu_{-\delta}$ and consisting of subarcs of the curves Γ_δ, Δ_γ, $\Gamma_{-\delta}$, Δ_μ. The curve $C_\delta(\mu, \gamma)$ is defined so as to contain in its interior the interior of the directed segment (μ, γ) on the oriented contour Γ_0. The integrand $f(\xi)$ in $I(\mu, \gamma)$ is defined and continuous at every point of $C_\delta(\mu, \gamma)$ except at μ and γ. In view of the condition (G_m), $f(\xi)$ is bounded on $C_\delta(\mu, \gamma)$. Hence $I(\mu, \gamma)$ exists and is clearly independent of δ since $f(\xi)$ has its only singularities on Γ_0. It will first be shown that $I(\mu, \gamma) \to 0$ as $|\mu - \gamma| \to 0$. Let $\epsilon > 0$ be arbitrary and fix $\delta > 0$ so that the arcs $\mu_{-\delta}\mu_\delta$ and $\gamma_{-\delta}\gamma_\delta$ both have length less than ϵ. Fix K_ϵ so that

$$|R(\xi; T)| = K_\epsilon, \qquad \xi \in \Gamma_\delta \cup \Sigma_{-\delta}$$

and fix $\alpha_\epsilon > 0$ so that the arcs $\gamma_{-\delta}\mu_{-\delta}$ and $\mu_\delta\gamma_\delta$ both have length less than ϵ/K_ϵ provided that $|\mu-\gamma| < \alpha_\epsilon$. Then, for $|\mu-\gamma| < \alpha_\epsilon$, we have

$$|I(\mu, \gamma)| \leqq \frac{1}{2\pi}[2\epsilon M + 2K_\epsilon\epsilon/K_\epsilon]$$

which shows that $I(\mu, \gamma) \to 0$ as $|\mu-\gamma| \to 0$.

It will next be observed that the spectrum of any vector of the form $I(\mu, \gamma)x$ is contained in the directed closed subarc $\mu\gamma$. To see this let λ lie in $\rho(T)$ and outside $C_\delta(\mu, \gamma)$. Then, it is seen from the resolvent equation that

$$R(\lambda; T)I(\mu, \gamma)x = \frac{1}{2\pi i}\int_{C_\delta(\mu,\gamma)} \frac{(\mu-\xi)^m(\gamma-\xi)^m}{\lambda-\xi} R(\xi; T)x\,d\xi$$

and the integral on the right of this equation gives an analytic extension of $R(\lambda; T)I(\mu, \gamma)x$ to all points outside of $C_\delta(\mu, \gamma)$. Thus the spectrum $\sigma(I(\mu, \gamma)x)$ is contained in the closed arc $\mu\gamma$. Thus if the closed intervals $\mu\gamma$ and $\mu'\gamma'$ of Γ_0 are disjoint we have from Lemma 2 of §3(A),

$$E(\mu\gamma)I(\mu', \gamma') = 0.$$

Let μ_n, μ, γ, γ_n be an ordered set on the oriented curve Γ_0 with $\mu_n \to \mu$ and $\gamma_n \to \gamma$. Then $E(\mu\gamma)I(\gamma_n, \mu_n) = 0$. Since

$$(\mu I - T)^m(\gamma I - T)^m = I(\mu_n, \mu) + I(\mu, \gamma) + I(\gamma, \gamma_n) + I(\gamma_n, \mu_n),$$

and $I(\mu_n, \mu) \to 0$, $I(\gamma, \gamma_n) \to 0$, it follows that

$$E(\mu\gamma)(\mu I - T)^m(\gamma I - T)^m = I(\mu, \gamma).$$

Now let Γ_0 be divided into n nonoverlapping subintervals $\sigma_1, \cdots, \sigma_n$ each of length L/n where L is the length of Γ_0, and let $\xi_1, \xi_2, \cdots, \xi_n$, $\xi_{n+1} = \xi_1$ be their positively ordered sequence of end points. Then, according to the preceding equation

$$\text{(viii)} \qquad E(\sigma_j)(\xi_j I - T)^m(\xi_{j+1} I - T)^m = I(\xi_j, \xi_{j+1}).$$

Now let $\delta = 1/n$ so that the length of $C_\delta(\xi_j, \xi_{j+1})$ is of the order of $1/n$. It follows from (vii) and (viii) that

$$E(\sigma_j)(\xi_j I - T)^m(\xi_{j+1} I - T)^m = \frac{C_1}{n^{m+1}},$$

and thus that

$$\left|\sum_{j=1}^{n} (\xi_j I - T)^m(\xi_{j+1} I - T)^m E(\sigma_j)\right| = \frac{C_2}{n^m}\cdot$$

An elementary continuity argument shows that the sum whose norm appears on the left of the preceding inequality approaches the integral $\int_{\sigma(T)}(\xi I - T)^{2m}E(d\xi)$ as $n \to \infty$ and thus, this inequality establishes the Equation (vi).

It will next be proved that

$$\text{(ix)} \qquad \int_{\sigma(T)} (\xi I - T)^j E(d\xi) = 0 \qquad j \geqq m.$$

In view of (vi) this equation may be proved by induction downward. Thus it will be shown that (ix) holds for the integer $j \geqq m$ provided that it holds for the integer $j+1$. To do this let ξ_δ be the point of intersection of the curve Γ_δ with the transversal Δ_ξ through the point ξ on Γ_0 and let $R(\xi_\delta) = R(\xi_\delta; T)$. Then, from (iv) and the hypothesis (G_m) we have

$$\Big|\int_{\sigma(T)} (\xi - \xi_\delta)^{j+1} R(\xi_\delta) E(d\xi) = C_1 \,|\, \xi - \xi_\delta \,|\,,$$

which shows that

$$\lim_{\delta \to 0} \int_{\sigma(T)} (\xi - \xi_\delta)^{j+1} R(\xi_\delta) E(d\xi) = 0.$$

On the other hand it is seen by writing $(\xi - \xi_\delta)I = (\xi I - T) - (\xi_\delta I - T)$ that

$$\int_{\sigma(T)} (\xi - \xi_\delta)^{j+1} R(\xi_\delta) E(d\xi) = \int_{\sigma(T)} (\xi_\delta I - T)^{j+1} R(\xi_\delta) E(d\xi)$$

$$+ \sum_{r=1}^{j+1} (-1)^r \binom{j+1}{r} \int_{\sigma(T)} (\xi I - T)^{j+1-r} (\xi_\delta I - T)^{r-1} E(d\xi).$$

But, in view of (v) and the induction hypothesis,

$$\int_{\sigma(T)} (\xi I - T)^{j+1} R(\xi_\delta) E(d\xi) = 0, \qquad 0 < \delta \leqq 1.$$

Thus, since $\lim_{\delta\to 0}(\xi_\delta I-T)^{r-1}=(\xi I-T)^{r-1}$ in the uniform operator topology it follows from (iv) that

$$\lim_{\delta\to 0}\int_{\sigma(T)}(\xi-\xi_\delta)^{j+1}R(\xi_\delta)E(d\xi)=-\int_{\sigma(T)}(\xi I-T)^jE(d\xi)$$

which establishes the equation (ix) for every $j\geqq m$. Now if S is the scalar part of T then

$$\begin{aligned}0=\int_{\sigma(T)}(\xi I-T)^mE(d\xi)&=\sum_{r=0}^{m}\binom{m}{r}\int_{\sigma(T)}\xi^{m-r}E(d\xi)(-T)^r\\&=\sum_{r=0}^{m}\binom{m}{r}S^{m-r}(-T)^r=(S-T)^m=(-N)^m,\end{aligned}$$

which shows that $N^m=0$. The fact that T is of type $m-1$ is now evident. Q.E.D.

The spectral theorem for a bounded selfadjoint operator T in Hilbert space follows from Theorem 18. A well known and elementary argument shows that the spectrum is real and that the resolvent has first order rate of growth along vertical lines. This shows that the condition (G_m) is satisfied with $m=1$. Now, since $|x+y|^2=|x|^2+|y|^2$ for orthogonal vectors, to verify the condition (B) it will suffice to show that x and y are orthogonal if their spectra are disjoint. In this case the function

$$(R(\lambda;T)x,y)=(x,R(\bar\lambda;T)y)=\operatorname{conj}(R(\bar\lambda;T)y,x)$$

the complex conjugate of $(R(\bar\lambda;T)y,x)$, is analytic for $\lambda\notin\sigma(x)$ and $\bar\lambda\notin\sigma(y)$. Since $\sigma(y)$ is real and disjoint with $\sigma(x)$ the function

$$(R(\lambda:T)x,y)=\frac{(x,y)}{\lambda}+\frac{(Tx,y)}{\lambda^2}+\cdots$$

is everywhere analytic and vanishes at infinity. Hence it is identically zero and the coefficient $(x,y)=0$. This verification of the condition (B) is due to Schwartz. Another one may be found in [12].

Another special case is useful in the discussion of the nonselfadjoint second order singular differential boundary value problems discussed by Neumark. Because of its special nature however, an independent proof, shorter than that of Theorem 18 is possible and will be given here. In this theorem the operator T has its spectrum in the smooth curve Γ_0 as described in the discussion of the growth condition (G). It will not be necessary however to assume the growth condition as it will follow from the other assumptions.

Theorem 20. *Let the operator T in the reflexive space $\mathfrak{X}$ have its spectrum in the curve Γ_0. In addition let $\mathfrak{X}_0$, $\mathfrak{X}_0^*$ be dense linear manifolds in the spaces $\mathfrak{X}$, $\mathfrak{X}^*$ respectively with the following three properties.*

(i) *For x_0 in $\mathfrak{X}_0$ and x_0^* in $\mathfrak{X}_0^*$ there is a constant $K(x_0^*, x_0)$ with*

$$|x_0^*R(\xi(t, \delta); T)x_0| \leqq K(x_0^*, x_0), \qquad -1 \leqq t, \delta \leqq 1, \delta \neq 0.$$

(ii) *For each x_0 in $\mathfrak{X}_0$ and x_0^* in $\mathfrak{X}_0^*$ the limits*

$$R^+(\lambda, x_0^*, x_0) = \lim_{\delta\to 0^+} x_0^*R(\xi(t, \delta); T)x_0,$$

$$R^-(\lambda, x_0^*, x_0) = \lim_{\delta\to 0^-} x_0^*R(\xi(t, \delta); T)x_0,$$

exist for each point $\lambda = \xi(t, 0)$ in Γ_0.

(iii) *There is a constant M depending only upon T such that*

$$\int_{\sigma(T)} |R^+(\lambda, x_0^*, x_0) - R^-(\lambda, x_0^*, x_0)|\, ds \leqq M|x_0^*|\,|x_0|, \qquad x_0 \in \mathfrak{X}, x_0^* \in \mathfrak{X}^*,$$

where s is the arc length on Γ_0.

Then T is a scalar type spectral operator whose spectral resolution is given by the formula

$$\text{(iv)} \qquad x_0^*E(\sigma)x_0 = \frac{1}{2\pi i}\int_\sigma \{R^+(\lambda, x_0^*, x_0) - R^-(\lambda, x_0^*, x_0)\}\, d\lambda.$$

Proof. Let $\delta > 0$ and let f be single valued and analytic in a neighborhood of the annular region bounded by the curves Γ_δ and $\Gamma_{-\delta}$. Then[10]

$$f(T) = \frac{1}{2\pi i}\left[\int_{\Gamma_\delta} f(\lambda)R(\lambda; T)d\lambda - \int_{\Gamma_{-\delta}} f(\lambda)R(\lambda; T)d\lambda\right],$$

where both integrals are taken in the positive sense. Then, in view of the hypotheses (i) and (ii), this formula may be written as

$$x_0^*f(T)x_0 = \frac{1}{2\pi i}\int_{\Gamma_0} f(\lambda)\{R^+(\lambda, x_0^*, x_0) - R^-(\lambda, x_0^*, x_0)\}\, d\lambda, \qquad x_0 \in \mathfrak{X}_0, x_0^* \in \mathfrak{X}_0^*,$$

or, since the integrand vanishes if λ is in the resolvent set,

$$\text{(v)} \quad x_0^*f(T)x_0 = \frac{1}{2\pi i}\int_{\sigma(T)} f(\lambda)\{R^+(\lambda, x_0^*, x_0) - R^-(\lambda, x_0^*, x_0)\}\, d\lambda, \qquad x_0 \in \mathfrak{X}_0, x_0^* \in \mathfrak{X}_0^*.$$

In view of the hypothesis (iii) the right side of this equation (v) defines, for each bounded Borel function f on $\sigma(T)$, a continuous bilinear form (f, x_0^*, x_0) which, since $\mathfrak{X}_0$ and $\mathfrak{X}_0^*$ are dense, has a unique extension to a bounded bilinear form (f, x^*, x) defined for all x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$. Since $\mathfrak{X}$ is reflexive it follows that there is a unique operator $f(T)$ in $\mathfrak{X}$ for which (v) holds. The mapping $f \rightarrow f(T)$ is a homomorphism on the algebra of analytic functions f and since every continuous function on Γ_0 is the uniform limit of analytic functions it follows that this map is also a homomorphism on the algebra of continuous functions. To see that it is a homomorphism on the algebra of bounded Borel functions note that for a fixed continuous function g the set of all bounded Borel functions f for which

$$(fg)(T) = f(T)g(T), \tag{vi}$$

includes all continuous functions. Furthermore, if the equation (vi) holds for each function in a uniformly bounded pointwise convergent sequence $\{f_n\}$ then it follows from (v) that it holds for the limit function $f = \lim f_n$. This shows that (vi) holds for every bounded Borel function f and every continuous function g. A repetition of this argument shows that it also holds if f and g are both bounded Borel functions. Thus the operators $f(T)$ and $g(T)$ commute and also, as (v) shows, satisfy the inequality

$$|f(T)| \leqq M \sup_{\lambda \in \sigma(T)} |f(\lambda)|. \tag{vii}$$

These facts show that the operator $E(\sigma)$ defined by (iv) is a bounded spectral measure which, in view of the Orlicz-Pettis theorem, is countably additive in the strong operator topology. To see that E is a spectral resolution for T it will therefore be sufficient to show that, for each Borel subset σ of $\sigma(T)$, the spectrum of the restriction $T|E(\sigma)\mathfrak{X}$ is contained in $\bar{\sigma}$. For every $\xi \notin \bar{\sigma}$ let the bounded Borel function r_ξ be defined by the equation $r_\xi(\lambda) = (\xi - \lambda)^{-1}\chi_\sigma(\lambda)$ where χ_σ is the characteristic function of σ. Then $r_\xi(T)(\xi I - T) = E(\sigma)$ which shows that $(T|E(\sigma)\mathfrak{X}) \subseteq \bar{\sigma}$, and proves that T is a spectral operator. It follows from (vii) that T is a scalar operator. Q.E.D.

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OUTLINE OF AN ALGORITHM FOR INTEGER SOLUTIONS TO LINEAR PROGRAMS

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The problem of obtaining the best integer solution to a linear program comes up in several contexts. The connection with combinatorial problems is given by Dantzig in [1], the connection with problems involving economies of scale is given by Markowitz and Manne [3] in a paper which also contains an interesting example of the effect of discrete variables on a scheduling problem. Also Dreyfus [4] has discussed the role played by the requirement of discreteness of variables in limiting the range of problems amenable to linear programming techniques.

It is the purpose of this note to outline a finite algorithm for obtaining integer solutions to linear programs. The algorithm has been programmed successfully on an E101 computer and used to run off the integer solution to small (seven or less variables) linear programs completely automatically.

The algorithm closely resembles the procedures already used by Dantzig, Fulkerson and Johnson [2], and Markowitz and Manne [3] to obtain solutions to discrete variable programming problems. Their procedure is essentially this. Given the linear program, first maximize the objective function using the simplex method, then examine the solution. If the solution is not in integers, ingenuity is used to formulate a new constraint that can be shown to be satisfied by the still unknown integer solution but not by the noninteger solution already attained. This additional constraint is added to the original ones, the solution already attained becomes nonfeasible, and a new maximum satisfying the new constraint is sought. This process is repeated until an integer maximum is obtained, or until some argument shows that a nearby integer point is optimal. What has been needed to transform this procedure into an algorithm is a systematic method for generating

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the new constraints. A proof that the method will give the integer solution in a finite number of steps is also important. This note will describe an automatic method of generating new constraints. The proof of the finiteness of the process will be given separately.

Let us suppose that the original inequalities of the linear program have been replaced by equalities in nonnegative variables, so that the problem is to find nonnegative integers, $w, x_1, \cdots, x_m, t_1, \cdots, t_n$, satisfying

$$\begin{array}{l} w = a_{0,0} + a_{0,1}(-t_1) \cdots a_{0,n}(-t_n), \\ x_1 = a_{1,0} + a_{1,1}(-t_1) \cdots a_{1,n}(-t_n), \\ \cdot \\ \cdot \\ \cdot \\ x_m = a_{m,0} + a_{m,1}(-t_1) \cdots a_{m,n}(-t_n) \end{array} \tag{1}$$

such that w is maximal. Using the method of pivot choice given by the simplex (or dual simplex) method, successive pivots result in leading the above array into the standard simplex form,

$$\begin{array}{l} w = a'_{0,0} + a'_{0,1}(-t'_1) \cdots a'_{0,n}(-t'_n), \\ x'_1 = a'_{1,0} + \quad \cdots \quad \cdots a'_{1,n}(-t'_n), \\ \cdot \\ \cdot \\ \cdot \\ x'_m = a'_{m,0} + \quad \cdots \quad \cdots a'_{m,n}(-t'_n) \end{array} \tag{2}$$

where the primed variables are a rearrangement of the original variables and the $a'_{0,j}$ and $a'_{i,0}$ are nonnegative. From this array the simplex solution $t'_j = 0$, $x'_i = a'_{i,0}$ is read out.

An additional constraint can now be formulated. The constraint which will be generated is not unique, but is one of a large class that can be produced by a more systematic version of the following procedure.

If the $a'_{i,0}$ are not all integers, select some i_0 with $a'_{i_0,0}$ noninteger, and introduce the new variable

$$s_1 = -f'_{i_0,0} - \sum_{j=1}^{j=n} f'_{i_0,j}(-t'_j) \tag{3}$$

where $f'_{i_0,j} = a'_{i_0,j} - n'_{i_0,j}$, with $n'_{i_0,j}$ the largest integer $\leqq a'_{i_0,j}$. This new equation is added to the Equations (2), obtaining a new set which will be referred to as (2*). A feasible solution to (2*) is a vector, $w', x'_1, \cdots, x'_m, t'_1, \cdots, t'_n, s_1$ of nonnegative components. The values of $x'_1, \cdots, x'_m, t'_1, \cdots, t'_n$ determine the s_1 value through (3), so there is a natural correspondence between a solution

$x_1', \cdots, x_m', t_1', \cdots, t_n'$ of (2) and the (not necessarily feasible) solution that these values determine for (2*). Clearly any feasible solution to (2*) determines a feasible solution to the equations (2) simply by dropping the s_1.

It should be noted that if $f_{i_0,0}$ is $\neq 0$, then there is at least one $f_{i_0,j} \neq 0$, with $j \neq 0$, otherwise the equation

$$x'_{i_0} = a'_{i_0,0} + \sum_{j=1}^{j=n} a'_{i_0,j}(-t'_j)$$

can have no solution in integers, and the program has no integer solution.

Since the simplex solution to (2), $t'_j = 0$, $x'_i = a'_{i,0}$ determines, through Equation (3), a negative value, $-f_{i_0,0}$ for s_1, the corresponding solution to (2*) is not feasible, i.e. the new restraint cuts off the old maximum. However, any nonnegative *integer* solution to (2) does give rise to a nonnegative integer solution to the equations (2*).

To see this suppose $w'', x_1'', \cdots, x_m'', t_1'', \cdots, t_n''$ is any solution in nonnegative integers to (2). The s_1'' determined is

$$\begin{aligned} s_1'' &= -f'_{i_0,0} - \sum_{j=1}^{j=n} f'_{i_0,j}(-t''_j) \\ &= n'_{i_0,0} + \sum_{j=1}^{j=n} n'_{i_0,j}(-t''_j) - a'_{i_0,0} - \sum_{j=1}^{n} a'_{i_0,j}(-t''_j) \end{aligned}$$

which using (2) becomes $s_1'' = n'_{i_0,0} + \sum_{j=1}^{n} n'_{i_0,j}(-t_j'') - x''_{i_0}$. Since the $n'_{i_0,j}$, the t_j'' and the x''_{i_0} are all integers, the s_1'' determined is also an integer. Furthermore, since the $f'_{i_0,j}$ and the t_j'' are all nonnegative, (3) shows that s_1'' is $\geqq -f'_{i_0,0} > -1$. Since s_1'' is an integer, this shows it must be nonnegative.

This reasoning establishes a one-one correspondence between nonnegative integer solutions $w'', x_1'', \cdots, x_m'', t_1'', \cdots, t_n''$ to (2) and the corresponding nonnegative integer solutions $w'', x_1'', \cdots, x_m'', t_1'', \cdots, t_n'', s_1''$ to (2*). Since the w value is the same for both solutions, the problem of maximizing w over nonnegative integer solutions to (2) can be replaced by the problem of maximizing w over the nonnegative integer solutions to (2*). The solution to the original problem is obtained by dropping the s_1.

The procedure now is to maximize w over the solutions to (2*). This is done using the dual simplex method because all the $a'_{0,j}$ and $a'_{i,0}$ are already nonnegative, and $-f_{i_0,0}$ is the only negative entry in the zero column of the equations (2*). This fact usually makes remaximization quite rapid. The process is then repeated if the new simplex maximum is noninteger.

Of course the Equations (2*) involve one more equation than the Equations (2), and an equation is added after each remaximization. However, the total number need never exceed $m+n+2$. For if an s-variable, added earlier in the computation reappears among the variables on the left hand side of the equations after some remaximization, the equation involving it can simply be dropped, as the only equations that need be satisfied are the original ones. This limits the total number of s-variables to $n+1$ or less.

It should be noted that even the process just described involves an element of choice, any of the rows i of (2) with $a_{i,0}$ noninteger might be chosen to generate the new relation. Some choices are better than others. A good rule of thumb based on the idea of "cutting" as deeply as possible with the new relation, and borne out by limited computational experience, is to choose the row with the largest fractional part $f_{i,0}$ in the zero column.

The class of possible additional constraints is not limited to those produced by the method described here since it is easily seen that some simple operations on and between rows preserve the properties needed in the additional relations. These operations can be used to produce systematically a family of additional relations from which a particularly effective cut or cuts can be selected. A discussion of this class of possible additional constraints together with a rule of choice of row which can be shown to bring the process to an end in a finite number of steps—thus providing a finite algorithm—require some space and will be given as part of a more complete treatment in another place.

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PRINCETON UNIVERSITY

ON THE NONEXISTENCE OF ELEMENTS OF HOPF INVARIANT ONE

BY J. F. ADAMS

Communicated by S. Eilenberg, April 29, 1958

With the usual definitions of homotopy-theory, we have the following theorem.

THEOREM 1. (a) S^{n-1} *is not an H-space unless* $n=2$, 4, *or* 8.

(b) *There is no element of Hopf invariant one in* $\pi_{2n-1}(S^n)$ *unless* $n=2$, 4, *or* 8.

For the context of this question, see [5] (especially pp. 436–438), [4, Chapter VI] and [6, §§20, 21].

This theorem results from reasonings with secondary cohomology operations. It is generally understood that a secondary operation corresponds to a relation between primary operations. One may formalize the notion of a "relation" by introducing pairs (d, z), algebraic in nature, as follows.

Let p be a prime; let A be the Steenrod algebra [2, p. 43] over Z_p. One defines the notion of a graded left module M over the graded algebra A so that $M=\sum_q M_q$ and $A_q M_r \subset M_{q+r}$. For example, let us write $H^q(X)$ for $H^q(X; Z_p)$, $H^*(X)$ for $\sum_q H^q(X; Z_p)$ and $H^+(X)$ for $\sum_{q>0} H^q(X; Z_p)$; then $H^*(X)$ and $H^+(X)$ are graded left modules over A. Let M, N be such modules; one defines the notion of an A-map $f: M \rightarrow N$ of degree r so that $f(M_q) \subset N_{q+r}$.

A pair (d, z), then, is to have the following nature. The first entry d is to be an A-map $d: C_1 \rightarrow C_0$ of degree zero. Here C_0, C_1 are to be modules in the above sense; we require, moreover, that they are locally finitely-generated and free, and that $(C_i)_q=0$ if $q<i$ $(i=0, 1)$. The second entry z is to be a homogeneous element of Ker d.

Let (d, z), then, be a pair of this sort. We call Φ a stable secondary cohomology operation associated with (d, z), if it satisfies the following axioms.

AXIOM (1). $\Phi(\epsilon)$ is defined for each A-map ϵ: $C_0 \rightarrow H^+(X)$ of degree $m \geqq 1$ and such that $\epsilon d=0$.

Such a map ϵ is determined by its values on the elements of an A-base of C_0. It therefore corresponds to a set of elements of $H^+(X)$. In particular, if C_0 is free on one given generator c, we write $u=\epsilon c$; we may thus consider Φ as a function of one variable u, where u runs over a subset of $H^+(X)$. In this case we write $\Phi(u)$ for $\Phi(\epsilon)$.

For the next axiom, set $\deg(z)=n+1$, let f: $C_1 \rightarrow H^+(X)$ run over the A-maps of degree $(m-1)$, and let $Q^{m+n}(d, z; X)$ be the set of elements of the form fz.

Axiom (2). $\Phi(\epsilon) \in H^{m+n}(X)/Q^{m+n}(d, z; X)$.

For the next axiom, let $g\colon Y \rightarrow X$ be a map.

Axiom (3). $g^*\Phi(\epsilon) = \Phi(g^*\epsilon)$.

For the next axiom, let (X, Y) be a pair, and let $\epsilon\colon C_0 \rightarrow H^+(X)$ be a map of degree $m \geqq 1$ such that $\epsilon d = 0$ and $i^*\epsilon = 0$. We can now form the following diagram.

$$\begin{tikzcd}
H^+(Y) & H^+(X) \arrow[l, "i^*"'] & H^+(X, Y) \arrow[l, "j^*"'] & H^+(Y) \arrow[l, "\delta"'] & H^+(X) \arrow[l, "i^*"'] \\
 & & C_0 \arrow[ul, "\epsilon"] \arrow[u, "\eta"'] & C_1 \arrow[l, "(-1)^m d"'] \arrow[u, "\zeta"'] &
\end{tikzcd}$$

Axiom (4). $i^*\Phi(\epsilon) = \{\zeta z\} \bmod i^*Q^{m+n}(d, z; X)$.

For the next axiom, let SX be the suspension of X, and let $\sigma\colon H^+(X) \rightarrow H^+(SX)$ be the suspension isomorphism. Let ϵ be as above.

Axiom (5). $\sigma\Phi(\epsilon) = \Phi(\sigma\epsilon)$.

Theorem 2. *Given any pair* (d, z) (*as above*), *there is at least one stable secondary cohomology operation* Φ *associated with it* (*in the sense of the axioms above*).

This theorem is proved by the method of the universal example. The next theorem allows us to study the operations Φ by applying homological algebra (see [3]) to the pairs (d, z).

Theorem 3. (a) *If* Φ, Φ' *are two operations associated with the same pair* (d, z) *then there is an element* c *in* $(C_0/dC_1)_n$ *such that*

$$\Phi(\epsilon) - \Phi'(\epsilon) = \{\epsilon c\}.$$

(b) *Suppose given* d (*as above*), *elements* z_t *in* Ker d, *and operations* Φ_t *associated with the pairs* (d, z_t). *Suppose* $z = \sum_t a_t z_t$ $(a_t \in A)$. *Then there is an operation* Φ *associated with* (d, z) *such that*

$$\sum_t a_t\Phi_t(\epsilon) = \{\Phi(\epsilon)\} \bmod \sum_t a_t Q^{m+n_t}(d, z_t; X).$$

(c) *Suppose given a diagram*

$$\begin{tikzcd}
C_1 \arrow[r, "m_1"] \arrow[d, "d"'] & C_1' \arrow[d, "d'"] \\
C_0 \arrow[r, "m_0"] & C_0'
\end{tikzcd}$$

in which d, d' *are as above, and* m_0, m_1 *are* A*-maps of degree zero. Let* Φ *be an operation associated with a pair* (d, z). *Then there is an operation* Φ' *associated with* $(d', m_1 z)$ *such that*

$$\Phi(\epsilon' m_0) = \{\Phi'(\epsilon')\}$$

for each $\epsilon' : C_0' \to H^+(X)$ *of the sort considered above.*

One may show that for operations in one variable, there is a Cartan formula for expanding $\Phi(uv)$, where uv is a cup-product.

We now take $p=2$. We also omit to summarize some work with homological algebra. This work leads us to consider certain pairs (d, z). By applying Theorem 2, we obtain secondary operations $\Phi_{i,j}(u)$ for $0 \leqq i \leqq j$, $j \neq i+1$. The operation $\Phi_{j,i}(u)$ is of degree 2^i+2^j-1, and it is defined on classes u such that $Sq^{2^r}(u)=0$ for $0 \leqq r \leqq j$.

Let P be complex projective space of infinitely-many dimensions, and let y be a generator of $H^2(P)$ (by which we mean $H^2(P; Z_2)$). We may ask for the values of the operations $\Phi_{i,j}$ in $H^*(P)$. Now, $\Phi_{i,j}(y^r)$ is defined only if $r \equiv 0$ and mod 2^j. Moreover, the degree of $\Phi_{i,j}$ is odd unless $i=0$ and $j>0$; so that $\Phi_{i,j}(y^r)$ lies in a zero group unless $i=0$ and $j>0$. It remains only to consider $\Phi_{0,j}(y^{n2^j})$; this is defined modulo zero.

THEOREM 4.

$$\Phi_{0,j}(y^{n2^j}) = ny^{(n+1/2)2^j} \qquad \text{(mod zero)}.$$

In the proof of this theorem we make essential use of a formula for the composite operation $\Phi_{0,j}Sq^{2^j}$. This formula is proved by applying Theorem 3.

THEOREM 5. *For each* $k \geqq 3$ *we have a formula*

$$\sum_{i,j;j\leq k} a_{i,j,k}\Phi_{i,j}(u) = Sq^{2^{k+1}}(u) \qquad \text{(mod } Q).$$

The formula is valid on classes u such that $Sq^{2^r}(u)=0$ for $0 \leqq r \leqq k$, and holds modulo a certain subgroup Q. It is proved as follows. By applying Theorem 3, we obtain a formula

$$\sum_{i,j;j\leq k} a_{i,j,k}\Phi_{i,j}(u) = \lambda Sq^{2^{k+1}}(u) \qquad \text{(mod } Q)$$

in which $a_{i,j,k} \in A$, and the coefficient λ remains to be determined. Applying the formula to a suitable class u in $H^*(P)$, we determine $\lambda = 1$.

To prove Theorem 1, it is sufficient to prove it for the case $n=2^m$. This case follows immediately from Theorem 5, using the same argument as that used by Adem [1, §4] in the case $n \neq 2^m$.

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INSTITUTE FOR ADVANCED STUDY

ON RAPIDLY MIXING TRANSFORMATIONS AND AN APPLICATION TO CONTINUED FRACTIONS

BY M. KAC[1] AND HARRY KESTEN

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1. Let Ω be a measure space (of total measure 1) and let T be a measure preserving transformation in the sense that

$$\mu\{T^{-1}(A)\} = \mu\{A\}, \tag{1.1}$$

where $T^{-1}(A)$ denotes the inverse image of A (we do not assume that T is necessarily one-to-one).

Let

$$V(\omega) = \begin{cases} 1, & \omega \in B, \\ 0, & \omega \notin B, \end{cases} \tag{1.2}$$

and consider

$$\sum_{k=1}^{n} V(T^k\omega). \tag{1.3}$$

It will be our purpose to sketch a proof of the theorem that under a suitable condition on T the "conditional" measure

$$\frac{\mu\left\{\dfrac{\sum_{k=1}^{n} V(T^k\omega)}{n} - \mu\{B\} < \dfrac{\alpha}{n^{1/2}},\ \omega \in B\right\}}{\mu\{B\}} \tag{1.4}$$

approaches as n tends to infinity

$$\frac{1}{\sigma(2\pi)^{1/2}}\int_{-\infty}^{\alpha} e^{-u^2/(2\sigma^2)}du, \tag{1.5}$$

where σ is, in general, not known explicitly.

The condition to be imposed on T is that of "exponentially rapid mixing" and can be stated as follows:

If ν and ν_m are defined by

$$\nu\{A\} = \frac{\mu\{A \cap B\}}{\mu\{B\}}, \qquad \nu_m\{A\} = \frac{\mu\{A \cap B \cap T^{-m}B\}}{\mu\{B \cap T^{-m}B\}}$$

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then for every measurable set A

$$(1.6)\quad \begin{aligned} &|\nu\{T^{-k}A\} - \mu\{T^{-k}A\}| = |\nu\{T^{-k}A\} - \mu\{A\}| \leqq He^{-\epsilon k}\mu\{A\}, \\ &|\nu_m\{T^{-k}A\} - \mu\{A\}| \leqq H_m e^{-\epsilon k}\mu\{A\}. \end{aligned}$$

$\epsilon > 0$ is an absolute constant, but H_m may vary with m.

This condition may seem more severe than Bernstein's (Math. Ann. vol. 97 (1927) pp. 1–59), but Bernstein's conditions would ask for a "uniformly rapid" decrease of all expressions

$$\left| \frac{\mu\{T^{-k}A \cap B_1 \cap \cdots \cap B_m\}}{\mu\{B_1 \cap \cdots \cap B_m\}} - \mu\{A\} \right|$$

where each $B_i = B$ or $\Omega - B$.

2. Let ν be defined as follows:

$$(2.1)\qquad \nu(A) = \frac{\mu(A \cap B)}{\mu(B)},$$

where B is a fixed set of positive measure. We set

$$(2.2)\qquad {}_{(k)}P_n^{(1)} = \frac{\int_\Omega V(T^k\omega) \exp\left(-u \sum_{r=k+1}^{k+n} V(T^r\omega)\right) d\nu}{\int_\Omega V(T^k\omega) d\nu},$$

$$(2.3)\qquad {}_{(k)}P_n^{(2)} = \frac{\int_\Omega (1 - V(T^k\omega)) \exp\left(-u \sum_{r=k+1}^{k+n} V(T^r\omega)\right) d\nu}{\int_\Omega (1 - V(T^k\omega)) d\nu}$$

(with the obvious convention $P_0 \equiv 1$) and

$$(2.4)\qquad S_k^{(q)} = \sum_{n=0}^{\infty} {}_{(k)}P_n^{(q)} z^n, \qquad q = 1, 2.$$

Setting also

$$B^{(1)} = B, \qquad B^{(2)} = \Omega - B \qquad V^{(1)} = 1, \qquad V^{(2)} = 0$$

we verify by an immediate calculation that

$$(2.5)\qquad \sum_q \nu\{T^{-k}(B^{(q)})\} S_k^{(q)} = 1 + z \sum_q e^{-uV^{(q)}} \nu\{T^{-(k+1)}(B^{(q)})\} S_{k+1}^{(q)},$$

where $\nu\{T^{-k}(B^{(q)})\}S_k^{(q)}$ is taken to be zero whenever $\nu\{T^{-k}(B^{(q)})\}=0$.

Multiplying both sides of (2.5) by z^k and summing on k from 0 to ∞ we obtain

$$S_0^{(1)} = \frac{1}{1-z} - (1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B^{(1)})\}S_k^{(1)}z^k. \tag{2.6}$$

It should, of course, be remembered that S depends on z and u. Dropping the superscript 1, we get, setting

$$Q(z) = \sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}(S_k - S_0)z^k, \tag{2.7}$$

$$S_0 = \frac{1}{(1-z)\left[1+(1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}z^k\right]} - \frac{(1-z)(1-e^{-u})Q(z)}{(1-z)\left[1+(1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}z^k\right]}. \tag{2.8}$$

3. Let us expand $(1-e^{-u})(1-z)Q(z)$ in a power series

$$(1-e^{-u})(1-z)Q(z) = \sum_{r=1}^{\infty}\gamma_r(u)z^r \tag{3.1}$$

(recall that (2.7) is not a power series expansion of Q since the S's themselves depend on z) and let us also write

$$\frac{1}{(1-z)\left[1+(1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}z^k\right]} = \sum_{r=0}^{\infty}\beta_r(u)z^r. \tag{3.2}$$

From (2.8) we have comparing coefficients

$$\begin{aligned}{}_{(0)}P_n &= \frac{\int_B \exp\left(-u\sum_{r=1}^{n}V(T^r\omega)\right)d\nu}{\int_B d\nu} \\ &= \frac{\int_B \exp\left(-u\sum_{r=1}^{n}V(T^r\omega)\right)d\mu}{\mu\{B\}} = \beta_n(u) + \sum_{r=1}^{n}\gamma_r(u)\beta_{n-r}(u),\end{aligned} \tag{3.3}$$

and hence setting $u = -i\xi/n^{1/2}$,

$$\begin{aligned}\frac{\int_B \exp\left(\frac{i\xi}{n^{1/2}}\left[\sum_1^n V(T^r\omega) - n\mu\{B\}\right]\right)d\mu}{\mu\{B\}} &= e^{-i\xi\mu\{B\}n^{1/2}}\beta_n\left(-\frac{i\xi}{n^{1/2}}\right) \\ &\quad + e^{-i\xi\mu\{B\}n^{1/2}}\sum_{r=1}^n \gamma_r\left(-\frac{i\xi}{n^{1/2}}\right)\beta_{n-r}\left(-\frac{i\xi}{n^{1/2}}\right).\end{aligned} \tag{3.4}$$

4. From condition (1.6) it follows immediately that

$$\sum_{k=1}^\infty \nu\{T^{-k}(B)\}z^k = \frac{\mu\{B\}z}{1-z} + R(z), \tag{4.1}$$

where $R(z)$ is analytic for $|z| < 1+\epsilon$.

It is now quite easily shown that

$$\lim_{n\to\infty} e^{-i\xi\mu\{B\}n^{1/2}}\beta_n\left(-\frac{i\xi}{n^{1/2}}\right) = e^{-\sigma^2\xi^2/2}, \tag{4.2}$$

where

$$\sigma^2 = \mu\{B\}[1-\mu\{B\}] + 2\mu\{B\}R(1). \tag{4.3}$$

It is harder to prove that

$$\lim_{n\to\infty} n \sup_{1\leq r\leq n}\left|\gamma_r\left(-\frac{i\xi}{n^{1/2}}\right)\right| = 0, \tag{4.4}$$

but once this is done one gets from (3.4) and (4.2) that

$$\lim_{n\to\infty}\frac{\int_B \exp\left(\frac{i\xi}{n^{1/2}}\left[\sum_1^n V(T^k\omega) - n\mu\{B\}\right]\right)d\mu}{\mu\{B\}} = e^{-\sigma^2\xi^2/2} \tag{4.5}$$

and the theorem announced in §1 follows.

It should be pointed out that in proving (4.4) one has to apply (1.6) to all ν_m.

5. The general theorem of this note was motivated by an application to continued fractions.

In this case Ω is the interval $[0, 1]$ and the invariant measure

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{dx}{1+x}\,. \tag{5.1}$$

The transformation Tx in question is given by the formula

$$Tx = \frac{1}{x} - \left[\frac{1}{x}\right] \tag{5.2}$$

and the crucial property (1.6) was proved by Paul Lévy [1].

Our result is thus that *the number of times a specified digit occurs among the first n digits in a continued fraction is asymptotically normally distributed.*

The method used in proving our main result was suggested by the occupation time problem for Markoff chains. The fuller discussion of this connection as well as that of a number of related results will appear elsewhere.

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CORNELL UNIVERSITY

REPRESENTATION OF ABSTRACT RIESZ POTENTIALS OF THE ELLIPTIC TYPE

BY A. V. BALAKRISHNAN

Communicated by Einar Hille, June 13, 1958

Using semigroup theory we are able to obtain an abstract definition of the Riesz potentials of the elliptic type as closed linear operators, as well as a representation for them without using continuation.

Let

$$T(\xi),\ \xi = (\xi_1, \xi_2, \cdots, \xi_n),\ -\infty < \xi_k < \infty,$$

be an n-parameter group (strongly continuous) of endomorphisms over a B-space X. Let

$$T(\xi) = \prod_{k=1}^{n} T_k(\xi_k),$$

each $T_k(\xi_k)$ being a strongly continuous one-parameter group with infinitesimal generator A_k. Let

$$C = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}A_iA_j,$$

where the matrix $[a_{ij}]$ is real symmetric and positive definite. Then C is the infinitesimal generator of a one-parameter strongly continuous semigroup $S(t)$, $0<t$, and further $\sup \|S(t)\| < \infty$. Now, in some previous work the author has shown that in such a case it is possible to define $(-C)^\alpha$, $\operatorname{Re} \alpha>0$, as closed linear operators, interpolating integral powers and having the semigroup property in α. Moreover, they have an explicit representation as Bochner integrals in terms of $S(t)$, which for $0<\operatorname{Re} \alpha<1$, is

$$(-C)^\alpha x = \frac{1}{\Gamma(-\alpha)}\int_0^\infty [S(t)x - x]t^{-\alpha-1}dt, \tag{1}$$

for $x \in D(C)$. Next, to simplify the notation, let

$$C = \sum_{i=1}^{n} A_i^2.$$

Then for every $x \in X$,

$$S(t)x = \frac{1}{(2(\pi t)^{1/2})^n}\int_{E_n} T(\xi)x \exp\left[-\sum_1^n \xi_k^2/4t\right] d\xi_1 \cdots d\xi_n. \tag{2}$$

The Riesz potentials are obtained by substituting (2) into (1). Thus:

$$(-C)^{\alpha}x = \frac{4^{\alpha}\Gamma(\alpha + n/2)}{\pi^{n/2}\Gamma(-\alpha)}\int_{E_n^+} [T(\xi)x + T(-\xi)x - 2x]\,|\,\xi\,|^{-2\alpha-n}dV_{\xi}$$

where

E_n is the 2^n-ant in which $\xi_k \geqq 0$ for all k,
$|\,\xi\,| = [\xi_1^2 + \cdots + \xi_n^2]^{1/2}$,
dV_{ξ} is the n-dimensional volume element.

With slight modification it is possible to write this as:

$$(-C)^{\alpha}x = \frac{4^{\alpha}\Gamma(\alpha + n/2)}{\pi^{n/2}\Gamma(-\alpha)(n + 2\alpha - 1)}\int_{E_n^+} [T(\xi) - T(-\xi)] \cdot\left[\sum_{1}^{n} \xi_k A_k x\right]|\,\xi\,|^{-2\alpha-n}dV_{\xi},$$

where the integral is to be taken in the Cauchy sense at infinity. In this latter form, for $\alpha = 1/2$ for instance, we get an abstract version of the conjugate transform in $L_p(E_n)$ spaces in its usual representation.

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ABSTRACT CAUCHY PROBLEMS OF THE ELLIPTIC TYPE

BY A. V. BALAKRISHNAN

Communicated by Einar Hille, June 13, 1958

Let A be the infinitesimal generator of a strongly continuous one-parameter semigroup $T(\xi)$, $0<\xi$, of endomorphisms over a B-space X. Suppose it is required to find a function $u(t)$, $0<t$, with values in X such that:

(i) $u(t)$, $u^1(t)$, $\cdots$, $u^{n-1}(t)$ are absolutely continuous, $u^k(t)$ being the derivative of $u^{k-1}(t)$.

(ii) $u^n(t)=(-1)^{n+1}Au(t)$.

(iii) $\|u^k(t)-u_k\|\to 0$, as $t\to 0+$, $k=0, \cdots, n-1$.

We call this an abstract Cauchy problem of the elliptic type (ACPE$_n$). We prove:

THEOREM 1. *The ACPE$_n$ has at most one solution provided*

$$(\mathrm{H}_1) \qquad \int_1^\infty \|T(\xi)\|\xi^{-\sigma-1}d\xi < \infty \text{ for every } \sigma > 0.$$

THEOREM 2. *Let $n=2$. Let the semi-group $T(\xi)$ satisfy* H$_1$ *and let $u(t)$ be any solution of the ACPE$_2$ such that*

$$(\mathrm{H}_2) \qquad \limsup_{t\to\infty} t^{-1} \operatorname{Log} \|u(t)\| \leqq 0.$$

Then necessarily

$$(1) \qquad u(t) = (t/2\pi^{1/2}) \int_0^\infty T(\xi)u_0\xi^{-3/2} \exp(-t^2/4\xi)d\xi.$$

A slightly different but useful version of Theorem 2 is:

THEOREM 3. *Let $n=2$. Let the semi-group $T(\xi)$ satisfy* H$_1$. *Let $u(t)$, $t>0$, satisfy* (i), (ii), *but* (iiia) *below in place of* (iii)

$$(\mathrm{iiia}) \qquad \|u(t) - u_0\| \to 0 \text{ as } t \to 0+.$$

Then, if $u(t)$ satisfies H$_2$ *in addition, $u(t)$ is again determined by* (1). *Moreover, if $\|T(\xi)\|\to 0$ as $\xi\to\infty$, then any such $u(t)$ has a similar property, viz.:*

$$\|u(t)\| \to 0, \text{ as } t\to \infty.$$

Results similar in principle have been obtained for other values of n.

As an application of these results we may consider the elliptic equation:

$$\frac{\partial^2}{\partial t^2} u(t, x) + a(x) \frac{\partial^2}{\partial x^2} u(t, x) + b(x) \frac{\partial}{\partial x} u(t, x) = 0, \tag{2}$$

where the functions $u(t, \cdot)$ are to lie in $C[\alpha, \beta]$, $-\infty \leqq \alpha < \beta \leqq \infty$, $a(x)$, $b(x)$ are continuous and $a(x) > 0$. It suffices to consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} u(t, x) = a(x) \frac{\partial^2}{\partial x^2} u(t, x) + b(x) \frac{\partial}{\partial x} u(t, x). \tag{3}$$

However, lateral conditions for semigroup solutions of (3) have been given by Feller and Hille. Theorem 3 thus yields, in particular, a corresponding result on the global boundedness of the solutions of (2) (the generalized Phragmén-Lindelöf principle of P. Lax) somewhat more general than obtained hitherto in that conditions on $a(x)$ and $b(x)$ are milder, being enough to insure semigroup solutions of (3).

UNIVERSITY OF SOUTHERN CALIFORNIA

THE JUNE MEETING AT CORVALLIS

The five hundred and forty-seventh meeting of the American Mathematical Society was held at Oregon State College in Corvallis, Oregon, on Saturday, June 21, 1958. There was a meeting of the Mathematical Association of America on Friday, June 20 and a meeting of the Society for Industrial and Applied Mathematics on Saturday afternoon. Among the 143 registrants at these meetings were 85 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Leo Sario of the University of California at Los Angeles addressed the Society Saturday morning on *Problems on Riemann surfaces.* He was introduced by Professor Edwin Hewitt.

There were three sessions for contributed papers with Professors R. A. Beaumont, Arthur Erdelyi and A. T. Lonseth presiding. Mr. Riggle was introduced by Professor Goheen, Mr. Hilzman by Professor Lonseth and Professor Schaefer by Professor Hacker.

R. S. PIERCE,
Acting Associate Secretary

BOOK REVIEWS

Mathematische Statistik. By B. L. van der Waerden. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957. 9+360 pp. DM 46. Bound DM 49.60.

The aim of this book is clearly stated in the preface. Since his student days the author has been consulted on statistical questions arising from various fields of application. By reflection and study of the literature he has learned better and better methods to deal with them. These methods are to be presented with realistic examples taken from natural and social sciences. Thus it is hoped that the reader will be spared some groping in the dark.

The style of the book is set with this object in view. A concrete and practical problem is posed. It is at once precisely and clearly formulated in mathematical terms. Frequently it is reduced to the calculation of a certain probability, that of the rejection of the null hypothesis when it is true. Whenever possible, the exact combinatorial solution is derived. Approximations are then given, either by Stirling's formula or expansion in series. This is sometimes helped out by graphing, by tabulating, and by reports on actual computations the author has undertaken. If a statistic is used in testing, its mean and variance are calculated and its asymptotic properties which are proved in the literature are cited. If a novel mathematical concept is involved, this is dragged out into the open and probed. When several methods are available, their merits are compared theoretically and/or empirically. For instance, at the end of the book it is concluded that Spearman's rank correlation is to be preferred to the newer one of M. G. Kendall.

This manner of proceeding makes it easy for the reader who wishes to learn about a particular topic with dispatch or the general layman who is discouraged by dross and jargon. Further pedagogic devices are used to facilitate the use of the book: some repetitions are allowed to make for greater independence between the chapters; each chapter begins with a summary of the required preparation; an index of examples classified according to subject matter; an English vocabulary (a tribute to the British-American school which coined the statistical nomenclature). The informal, sometimes personal style also helps. On p. 175, e.g., there is the suggestion that the company, between the choice of two statistics, may be more interested in saving money than improving its product. Read in German, where "money" is "Geld," I found this truism rather amusing. The mathematical level is somewhat below that of Cramér's Mathematical Methods of Statistics. For example, most asymptotic results are stated without proof. The first few chapters, alternating probability background with statistical applications, instead of piling up all mathematical tools at the beginning, succeed in breaking the usual monotony.

The contents will be given with a somewhat random sampling of particulars. Chapter 1, *General foundations* (with the subheading: "The study of this chapter is indispensable"). Of course it is the rigorous set-theoretic foundation that is given. Chapter 2, *Probability and frequency*: including test of the true p, hypergeometric distribution, the chi-square test. Chapter 3, *Mathematical tools*: including orthogonal transformation, the beta and gamma functions. Chapter 4, *Empiric determination of distribution, mean and variance.* This begins with a childhood reminiscence of picking 89 willow leaves to form a Quetelet curve; includes the recent result of Birnbaum and Tingey, a brief discussion of the Kolmogorov test, and references to the literature on order statistics. Chapter 5, *Fourier integral and limit theorems.* It is a good feature of the book that the second limit theorem (convergence of moments), the usefulness of which is amply illustrated both here and elsewhere, is given in addition to the Lévy-Cramér theorem. Chapter 6, *Gauss's theory of errors and Student's test.* Chapter 7, *The method of least squares*: with an example from the author's study of a Byzantine solar chart. Chapter 8, *Estimation of unknown parameters*: including the maximum likelihood method; Fréchet inequality (listed in the English vocabulary as Cramér-Rao, also as "information inequality"); sufficient statistic. Chapter 9, *Evaluation of observed frequencies.* Chapter 10, *Bio-assay*: including graphic methods and the recent stochastic approximation method of Robbins and Monro. Chapter 11, *Testing hypothesis*: including analysis of variance and the Neyman-Pearson theory. Chapter 12, *Order tests.* The Smirnov test is briefly discussed; Wilcoxon's test and the author's X test are discussed at some length. Chapter 13, *Correlation.* Chapter 14, *Tables.* There are 13 covering 19 pages.

One may of course question the choice of material. The emphasis on methods dealing with practical problems may have cut down the discussion of more stilted (and sometimes stultified) topics. It is explicitly stated in the Preface that completeness is not striven for. "Many important theories such as those of sequential tests, decision functions and stochastic processes are completely omitted." One may yet wonder if some brief introduction to these theories could not be included, leading from simple examples to some of the basic ideas behind them, with references to the literature for details, just as is done for other topics favored in the book. It would seem that some of the minor problems and methods included might gracefully make room for this. Even the utilitarian reader would have profited by some contact with these modern notions and been spared the labor of tackling the larger works to which the author deferred.

The reviewer takes exception to the remark on p. 98 that the strong law of large numbers "scarcely plays a role in mathematical statistics." This is like saying e.g. that Dedekind cut scarcely plays a role in numerical analysis (or dynamics). The point is, even if the strong law is meaningless in a final statistical statement it may well enter into an argument or proof which is essential to the statistical conclusion, just as the real number system is surely at the back of many calculations although the IBM machines yield nothing but terminating decimals. To cite one concrete example, the asymptotic normality result mentioned on p. 220 has been recently extended (albeit slightly) by using convergence with probability one.

Finally, irrelevantly but inevitably, a reader of Professor van der Waerden's new book cannot help recalling his well-known volumes on *Moderne Algebra*. The format is there complete with the graphic guide; the masterful exposition is there; and the various pedagogic devices mentioned above are there. If the total impression is different this is due more to the subject matter than to the treatment. Mathematical Statistics, being a branch of fiercely applied mathematics with a relatively short history, does not have nor perhaps even care for the idealism and formalism of Algebra. Indeed, the criteria of excellence are somewhat different in these two fields. Statistics is primarily concerned with utility, not beauty; nevertheless there is no lack of neat things in this volume, and a good deal more in the field, as there always will be when competent hands work with Mathematics.

K. L. CHUNG

Convexity. By H. G. Eggleston. Cambridge Tracts in Mathematics and Mathematical Physics, no. 47, Cambridge University Press, 1958. 8+136 pp. $4.00.

This tract provides a brief and clear introduction to the theory of convex sets in E^n on an elementary level. It is not intended for the specialist, because it covers in the main only topics found in Bonnesen and Fenchel's *Theorie der konvexen Körper*. There are, of course, innovations in methods and proofs. As examples we mention the greater use of duality in E^n and the proof that the mixed volumes are non-negative.

The scope of the book will become apparent from a brief survey of its contents. After treating the basic notions like convex cover, supporting and separating planes, convex polyhedra (called polytopes), the tract turns in Chapter 2 to Helly's and Carathéodory's theorems, their interrelation and some generalizations. Convex functions, in particular distance and supporting functions follow (Chapter 3). Then distance for convex sets, Blaschke's Selection Theorem and the approximation of a convex set by polyhedra and regular convex sets are discussed. Chapter 5 deals with linear and concave families of convex sets, mixed volumes, Steiner's symmetrization, the Brunn-Minkowski theorem (Brunn is spelled as Brünn throughout), and Minkowski's inequalities. The more general inequalities of Alexandrov-Fenchel are stated without proof.

In Chapter 6 we find some extremal problems like the isoperimetric problem, and the relations between the inradius, circumradius and the diameter, also Besicovitch's result on the asymmetry of plane convex sets. Sets of constant width are the topic of the concluding chapter.

The book will well serve its purpose of providing an introduction to the field for "those starting research and for those working on other topics who feel the need to use and understand convexity."

HERBERT BUSEMANN

The numerical solution of two-point boundary problems in ordinary differential equations. By L. Fox. New York, Oxford University Press, 1957. 11+371 pp. $9.60.

If insistence on deductive reasoning is one of the characteristics of mathematics, the numerical solution of mathematical problems is not a branch of pure mathematics. In its methods and spirit it is more closely related to the applied sciences, in that incomplete induction based on experimental evidence is the ultimate criterion, even though a good measure of theoretical analysis is indispensable for guidance and interpretation. It is true that there are numerical procedures whose theory is so well understood that a result can be obtained which is safely embedded between double numerical inequalities. However, at this time, and for the forseeable future, the number of practically important problems in this comfortable class is, and will remain, depressingly small. A good specialist in the art of computation should therefore be able to resist the mathematician in him, who might lure him into the ideal realm of pure analysis, away from his concrete problems, without, on the other hand, losing the incentive of availing himself of all the mathematics, highbrow or lowbrow, that can be put in the service of computation.

The author of this book is steering a safe course between these two dangerous rocks, somewhat closer, perhaps, to the second than to the first. He is steeped in the British tradition of Numerical Analysis, which was developed before the rise of high-speed computing machines. Most of the content of the book is applicable to machine computation as well, but, on the whole, more reliance is put on the facility of the human computer for quick scanning and flexible independent judgments than on speed of operation.

The mode of presentation is so careful and leisurely that any computer with a bachelor's degree should find it easy reading. Every variant of procedure is illustrated by numerical examples worked out in great detail. Also, many of the mathematical tools, such as the necessary facts about finite differences, the solution of linear algebraic systems and eigenvalue theory are fully explained, if not always proved. These features account for the surprising length of a book devoted to a rather special subject.

A boundary value problem for differential equations can be solved either by replacing the full problem by a finite difference, i.e. a matrix, problem or by solving a suitable sequence of initial value problems at one point until the particular solution that also satisfies the conditions at the other end point has been approximated with satisfactory accuracy. Both methods are explained, but much more space is devoted to the former.

The author believes that the mesh length should always be kept fairly large, and that the truncation error should be reduced by employing refined difference approximations. In order to cope with the higher difference terms, he recommends that the first order approximating equation be solved first, and that the higher correction terms be then introduced and taken care of by some iterative scheme.

After two introductory chapters on *Finite differences* and *The solution of algebraic equations*, the author turns to second order boundary value problems. Surprisingly, the author considers the solution of first order equations, which are taken up next, a more subtle matter than the second order problems. The new difficulties stem from the fact that the first approximating difference equations on which the calculations are based involve central differences over two mesh lengths, and are therefore of second order. The extraneous solutions of the difference equation introduced in this manner may affect the stability of the procedure. Various techniques for controlling these unstable fluctuations are described in the book.

Next, differential equations of higher order are discussed, as well as systems of equations. After that, there is a long chapter on eigenvalue problems. The presentation becomes more condensed in the following chapter devoted to *Initial value techniques for boundary-value problems.*

A special chapter deals with the accuracy of procedures introduced previously. In view of the difficulty of this subject the conclusions here are largely qualitative statements supported by many numerical examples. In the final chapter several topics are touched upon very briefly, such as discontinuous coefficients, Richardson's deferred approach to the limit, collocation, nonlinear eigenvalue problems, etc.

Inasmuch as computation is an art rather than a science it cannot be taught systematically. This book tries to transmit some of the skill, flexibility and flair which a good computer needs in addition to a knowledge of the standard methods. Such a mode of presentation tends to make a book somewhat repetitious and a little exhausting for consecutive reading. This is unavoidable and is not meant as a criticism. Once the reader has become familiar with the basic techniques by studying, say, the first 90 pages, he will find the rest of the book an extremely valuable reference for advice and help, whenever he has to solve boundary value problems for ordinary differential equations.

WOLFGANG WASOW

Problems in Euclidean Space: Application of convexity. By H. G. Eggleston, Pergamon Press, New York 1957. 8+165 pp. $6.50.

The book is grouped around ten principal, essentially unrelated problems. The author motivates collecting these problems in a book by their implicit or explicit connection with convexity. Actually, there is a much stronger bond between them, namely the spirit of the arguments, which is that of Besicovitch: the problems are special, the proofs require highly ingenious and intricate arguments, but use only the most elementary tools, euclidean geometry and trigonometry besides some simple facts on point sets and measure in E^2 or E^3. Most of the work is due to the author, some to others, in particular to Besicovitch.

The first problem concerns the characterization of a nonvoid open set in E^2 which is the intersection of a decreasing sequence of open connected sets. The second is Ulam's problem, whether a homeomorphism $(x, y) \rightarrow (x', y')$ of E^2, or of a square in E^2, on itself can be approximated by homeomorphisms in which only one variable is changed. Problem 3 deals with sets E of finite linear Hausdorff measure. Denote by E_α the projection of E on a line with direction α

and put $\nu(E)=\inf_\alpha H(E_\alpha)$. Then $\nu(E)<2\pi^{-1}H(E)$ for any measurable E, $\nu(E)\leqq 2^{-1}H(E)$ for connected E and $\nu(E)\leqq H(E)/(\sec\alpha+2\tan\alpha+\pi-4\beta-2\alpha)$, where $\tan\beta=2^{-1}\sec\alpha$ and $2^{-1}+\sin\alpha = 4\cos^2\alpha\cdot(1+4\cos^2\alpha)^{-1}$. The proof of these three inequalities takes 37 pages.

Problem 4 is Borsuk's problem in E^3, whether a set of diameter δ is always the union of four sets of diameter less than δ. The author prefers his involved solution as possibly generalizable to $n>3$ to Grünbaum's simple solution as a "lucky fluke."

The remaining problems concern bounded closed convex sets in E^2. Denote by $\delta(X, Y)$ the area of $X\cup Y-X\cap Y$ and put $T(X, n)=\inf\delta(X, P_n)$, where P_n traverses all convex polygons with at most n sides. Problem 5 considers several functions of this type, but the principal result is the convexity of $T(X, n)$ as function of n.

The other problems are easy to describe briefly: 6. Extremal problems for convex sets solved by triangles, 7. The asymmetry of sets with constant width, 8. Sets of constant width contained in a set of given minimal width, 9. Extremal properties of circumscribed triangles, 10. The closest packing of equilateral triangles.

Clearly, this book will not appeal to some, but will be most delightful to others. In any case, it proves the strength and vitality of plane geometry and should help to revise the attitude of those numerous educators in the U.S.A. who want to all but eliminate plane geometry from high school and college curricula.

HERBERT BUSEMANN

Die Lehre von den Kettenbruchen. Vol. II. *Analytisch-funktionentheoretische Kettenbruche.* By Oskar Perron. 3d ed. Stuttgart, Teubner, 1957. 6+316 pp. DM 49.

Earlier editions of this text on continued fractions contained in a single volume a part devoted to the arithmetic theory and a part devoted to function-theoretic aspects (analytic theory). In the third edition these two parts have been published in separate volumes. Volume I was published in 1954 and the publication of the present book, Volume II, marks the completion of the third edition. Because the second edition is well known and has served as a standard reference since its appearance in 1929, only the major changes occurring in Volume II are noted below.

Chapter I includes new sections on continued fractions with prescribed approximants (§3) and on a formula of Ramanujan (§8). Chapter II contains additional sections on new convergence critieria, including the parabola theorem, (§17), and on the convergence of the Ramanujan continued fraction (§16). Several theorems on convergence of continued fractions with positive and with real elements (§11 and §12) are deleted, the statement of two theorems is improved in §14, and §15 has been completely rewritten to unify the presentation of the Van Vleck-Jensen theorems with recent extensions of these theorems. Chapter III has new sections on C-fractions (§21) and some other expansions of power series into continued fractions (§31), and on the relation to J-fractions of polynomials whose zeros have negative real parts. The discussion of periodic continued fractions (§22) has been modified to include C-fractions, and parametric representation of two continued fraction transformations has been added (§26). Chapter IV has a new section on complete convergence (§38) which permits the statement of a necessary and sufficient condition for a determinate Hamburger moment problem in §39. Several sufficient conditions for determinate Stieltjes and Hamburger moment problems are deleted from §39. Chapters V and VI have been reproduced without essential change.

The stated objective of the book is to give in an easily intelligible way the present state of knowledge of the subject. The author has been confronted with the difficult task of selecting and coordinating the material of major importance and not all readers will agree with his selections. Any defects of the book are those of omission. The reviewer regrets the omission of the methods and viewpoint of positive definite continued fractions and, in particular, positive definite J-fractions. However, the numerous virtues of the book, among which are clarity of presentation, systematic citing of origins of theorems, and the many examples and formulas, will make it a valuable reference for many years to come.

W. T. Scott

Combinatorial topology. Vol. 2. *The Betti groups*. By P. S. Aleksandrov. Trans. by Horace Komm. Rochester, Graylock Press, 1957. 11 +244 pp. $6.50.

The original Russian edition of Aleksandrov's *Kombinatornaya topologiya* (Moscow-Leningrad, OGIZ, 1947) is a single volume consisting of five parts. The English translation of the first two parts has been published as Vol. 1 (See the Review in this Bulletin, Vol. 62, 1956, pp. 629–630). The present Vol. 2 is the translation of Part III (Chaps. VII–XII), which is devoted to homology and cohomology groups of locally finite abstract cell complexes and homology groups of compact metric spaces. It deals mainly with the construction of these groups and the proof of their topological invariance.

The entire book is intended as an introduction to the classical homology theory. As such, this is an excellent textbook with many good features. Within the author's scope, the treatment is unusually thorough. The material is well organized, and presented with great detail at a leisurely pace. There is a constant emphasis on geometric content. Compared with the modern trend, the use of algebraic machinery is rather limited. The exposition is supported by ample illustrations, examples, and figures. This should be most helpful to the reader in understanding the various concepts and results.

The beginning (Chap. VII) of this Vol. 2 introduces the auxiliary algebraic apparatus (chains, boundary operator). The homology groups of locally finite abstract cell complexes are studied in Chap. VIII. Here as in the later chapters, the coefficient domains considered are non-topologized Abelian groups and fields. The cohomology groups for abstract cell complexes are introduced in Chap. IX, where one finds a careful study of canonical homology and cohomology bases. These are immediately used to derive the relations among homology and cohomology groups over different coefficient domains. The proof, using simplicial approximations, of the invariance of the homology groups of polyhedra is given in Chap. X. In Chap. XI, a homology theory for compact metric spaces is developed and is based on the notion of proper cycle, which is of Vietoris type. By using the topological invariance of the homology groups of a compact metric space, a second proof of the invariance of the homology groups of polyhedra is obtained. The last Chap. XII studies relative cycles and local homology groups, with application to homology dimension and pseudomanifolds with boundary. This Vol. 2 closes with an appendix containing the required basic facts on Abelian groups.

The translation is remarkably smooth. This reviewer sincerely hopes for early publication of the English translation of the remaining Parts IV and V, where one will find important topics and applications of classical homology theory (Alexander-Pontrjagin duality theorem, an introduction to the theory of intersection, mappings of polyhedra and Lefschetz-Hopf fixed-point formula). Without studying these, a student reading the present Vol. 2 would be unable to grasp the full significance of the theory. On the other hand, the beginning graduate student will find this Vol. 2 an excellent introduction to material treated from a more advanced and modern point of view in Eilenberg-Steenrod's *Foundations of algebraic topology*, Princeton, 1952.

KY FAN

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